Delay-based Scheduling to Create Service Differentiation in Multiclass Queues

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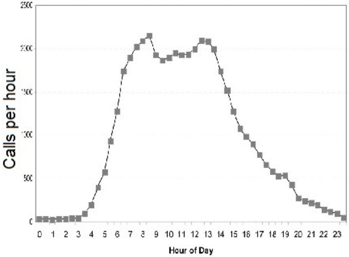
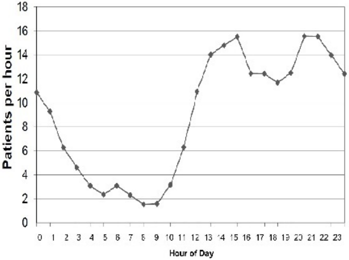
Time-Varying Arrival Rates





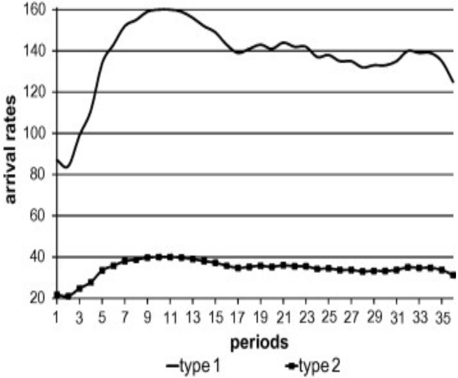
call center

Green et al. (2007)

emergency room

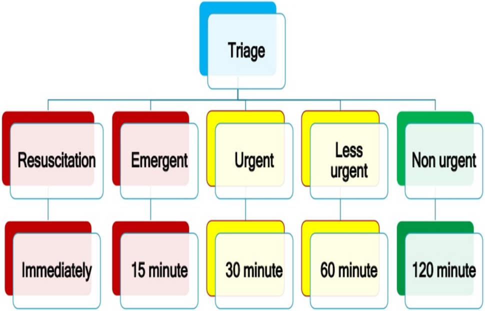
Yom-Tov and Mandelbaum (2011)

Differentiated Service Levels



- 80% of type 1 calls need to be answered within 20 seconds ("80-20 rule")
- 50% of type 2 calls need to be answered within 60 seconds
- How many servers are needed over the course of day?
- How to assign a newly idle agents to one of these queues?

Differentiated Service Levels



Canadian triage and acuity scale (CTAS, Ding et al. 2018)
"CTAS level i patients need to be seen by a physician within w_i minutes $100\alpha_i\%$ of the time", with

$$(w_1, w_2, w_3, w_4, w_5) = (0, 15, 30, 60, 120),$$
$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0.98, 0.95, 0.9, 0.85, 0.8).$$

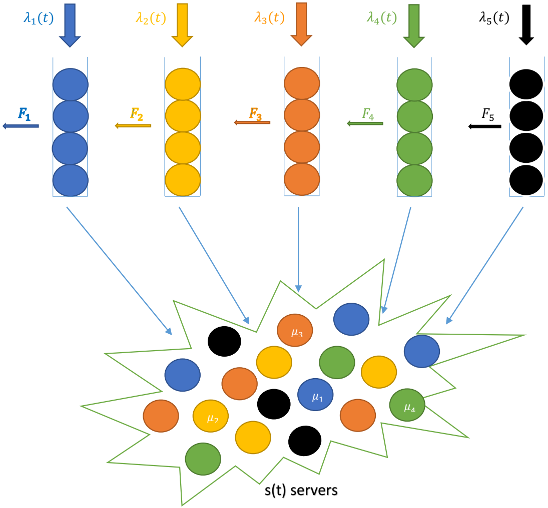
Other Examples of Service Differentiation

Existing Works and Contributions

- Literature review
  - Service differentiation
    Gurvich, Armony & Mandelbaum (08); Gurvich & Whitt (10); Soh & Gurvich (16); Kim, Randhawa & Ward (2017)
    All assume a critical-loading system and the demand to be stationary
  - Performance stabilization of time-varying queues
    Jennings et al. (96); Feldman et al. (08); Pender & Massey (17); Liu & Whitt (12,14); Liu (18)
    All consider single-class models

- Our contribution:
  studying service differentiation with time-varying demand and class-dependent services, focusing on overloaded systems.

A Multi-Class V Model



- Class-dependent arrival rate $\lambda_i(t)$ (non-homogeneous Poisson)
- Class-dependent abandonment-time distribution F_i
- A large time-varying number of servers $s(t)$
- Exponential service times with class-dependent service rate μ_i
- First-Come First-Served within each class

Problem Statement

- Model parameters

$$\mathcal{P} \equiv (\underbrace{\lambda_i(t), F_i, \mu_i,}_{\text{customer behavior}} \quad \underbrace{w_i, \alpha_i,}_{\text{service level}} \ 1 \le i \le K, \ 0 \le t \le T)$$

- Obtain convenient staffing and scheduling rules (in terms of $\mathcal{P}$), such that the *tail probability of delay* (TPoD)

$$\mathbb{P}\left(W_i(t) > w_i\right) \le \alpha_i, \qquad 1 \le i \le K, \quad 0 \le t \le T,$$
$$\text{or} \quad \mathbb{P}\left(W_i(t) > w_i\right) \approx \alpha_i$$

for any
  - $w_i > 0$ (delay target).
  - $\alpha_i \in (0,1)$ (probability target: fraction of excessive delay).

$W_i(t)$: potential waiting time of class i at time t, i.e., offered delay to a class-i arrival at t assuming infinitely patient.

Step I: Proposed Staffing Formula

1. Use offered-load (OL) to determine the nominal service capacity
   - No. of busy servers $B(t)$ in $M_t/GI/\infty \sim$ Poisson r.v. with mean

$$m(t) \equiv \mathbb{E}[B(t)] = \int_0^t \lambda(t-s) G^c(s) \mathrm{d}s.$$

   - Here, for the i^{th} class, the OL is

$$m_i(t) = \int_0^t \underbrace{F_i^c(w_i)\lambda_i(u-w_i)}_{\text{effective arr. rate}} \underbrace{e^{-\mu_i(t-u)}}_{\text{exp. service dist.}} \mathrm{d}u.$$

   OL: mean No. of busy servers needed to serve all customers who are willing to wait (excluding an acceptable faction of abandonment).

2. A **time-varying square-root staffing** (TV-SRS) rule

$$\boxed{s(t) = \underbrace{m(t)}_{\text{first order}} + \underbrace{\sqrt{\lambda^\star} c(t)}_{\text{second order}}} \quad \text{for} \quad m(t) \equiv m_1(t) + \cdots + m_K(t)$$

where $c(t)$ is a control function (TBD), and $\lambda^\star$ is the system's scale, i.e.,

$$\lambda^\star \equiv \frac{1}{T}\int_0^T \lambda(t)\mathrm{d}t, \qquad \text{with} \qquad \lambda(t) \equiv \lambda_1(t) + \cdots + \lambda_K(t).$$

Step I: Proposed Scheduling Structure

- Use real-time class-i head-of-line waiting time (HWT) $H_i(t)$ to devise a dynamic control policy;

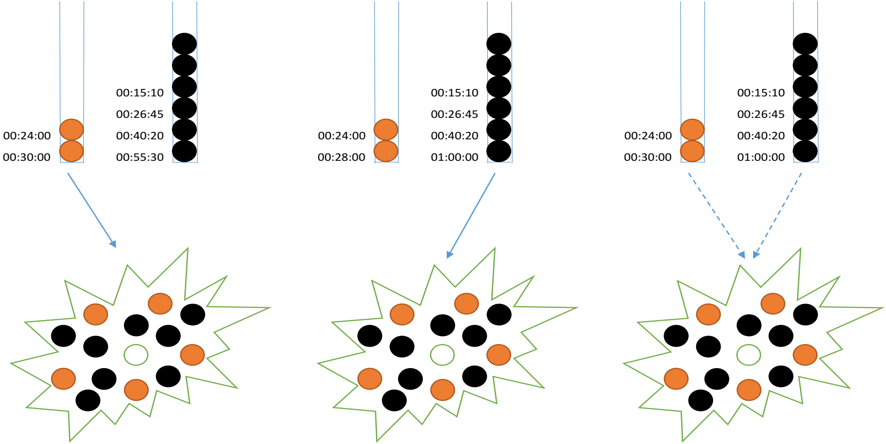
- A delay-based **time-varying dynamic prioritization scheduling** (TV-DPS) rule: Assigns the next available server to the HoL customer from class i^* satisfying

$$i^* \in \arg\max_{1 \le i \le K} \left\{ \underbrace{\frac{H_i(t)}{w_i}}_{\textbf{normalized HWT}} + \underbrace{\frac{1}{\sqrt{\lambda^\star}} \kappa_i(t)}_{\textbf{second order}} \right\}.$$

where $\kappa_i(t)$ is a control function (TBD).

- Main ideas of TV-DPS:
  - $\tilde{H}_i(t) \equiv H_i(t)/w_i$ focuses on the delay target w_i;
  - $\kappa_i(t)$ helps accomplish the class-dependent probability target α_i;
  - TV-DPS is both time-dependent (accounting for time variability) and state-dependent (capturing stochasticity).

An Illustration of How TV-DPS Rule Works



Goal: HoL delay ratio 1/2

Step II: Large-Scale Asymptotic Analysis

- Exact analysis is difficult; hence do asymptotic analysis as scale grows (realistic for large-scale systems).
- Use n in place of $\lambda^\star$ and consider a sequence of models indexed by n.
- In the n^{th} model:
  - Arrival rate $\lambda_i^n(t) \equiv n\lambda_i(t)$;
  - Staffing level:
  $$s^n(t) = nm(t) + \sqrt{n}c(t).$$
  - Scheduling rule:
  $$i^* \in \arg\max_{1 \le i \le K} \left\{ \frac{H_i^n(t)}{w_i} + \frac{1}{\sqrt{n}} \; \kappa_i(t) \right\}.$$
  - Service rates and abandonment distributions are fixed.
- HWT and PWT processes:
$$\widehat{H}_i^n(t) \equiv n^{1/2}\left(H_i^n(t) - w_i\right) \quad \text{and} \quad \widehat{W}_i^n(t) \equiv n^{1/2}\left(W_i^n(t) - w_i\right).$$
(Scaled "errors" of HWT and PWT)

Limit of Waiting Times and State-Space Collapse

Under the TV-SRS and TV-DPS policy, the CLT-scaled waiting time processes

$$\left(\widehat{H}_1^n, \ldots, \widehat{H}_K^n, \widehat{W}_1^n \ldots, \widehat{W}_K^n\right) \Rightarrow \left(\widehat{H}_1, \ldots, \widehat{H}_K, \widehat{W}_1 \ldots, \widehat{W}_K\right) \quad \text{in} \quad \mathcal{D}^{2K} \quad \text{as} \quad n \to \infty,$$

with all HWT and PWT limits in terms of a one-dimensional process $\widehat{H}(\cdot)$, where

$$\widehat{H}_i(t) \equiv w_i(\widehat{H}(t) - \kappa_i(t)), \qquad \widehat{W}_i(t) = w_i(\widehat{H}(t + w_i) - \kappa_i(t + w_i)).$$

The process $\widehat{H}$ uniquely solves the following *stochastic Volterra equation* (SVE)

$$\widehat{H}(t) = \int_0^t L(t,s)\widehat{H}(s)ds + \int_0^t J(t,s)d\mathcal{W}(s) + K(t),$$

where $\mathcal{W}$ is a standard Brownian motion,

$$L(t,s) \equiv \frac{\sum_{i=1}^K \eta_i(s)e^{\mu_i(s-t)}\left(\mu_i - h_{F_i}(w_i)\right)}{\eta(t)}, \quad J(t,s) \equiv \frac{\sqrt{\sum_{i=1}^K e^{2\mu_i(s-t)}\left(F_i^c(w_i)\lambda_i(s-w_i) + \mu_i m_i(s)\right)}}{\eta(t)},$$

$$K(t) \equiv \frac{\sum_{i=1}^K \left(\eta_i(t)\kappa_i(t) - \int_0^t \eta_i(s)e^{\mu_i(s-t)}\left(\mu_i - h_{F_i}(w_i)\right)\kappa_i(s)ds\right) - c(t)}{\sum_{i=1}^K \eta_i(t)}.$$

for $\eta_i(t) \equiv w_i\lambda_i(t - w_i)F_i^c(w_i)$.

Limit of Waiting Times and State-Space Collapse

1. State-space collapse (SSC): All $2K$ HWT and PWT processes degenerates to a one-dimensional process $\widehat{H}$.

2. The SVE admits a unique (strong) solution which is a Gaussian process
   - If $\mu_i \neq \mu$
     - ⋆ SVE has NO analytic solution;
     - ⋆ We gave effective algorithms (geometrically fast) to compute $m_{\widehat{H}}(t) \equiv \mathbb{E}[\widehat{H}(t)]$ and variance $C_{\widehat{H}}(t,s) \equiv \text{Cov}(\widehat{H}(t), \widehat{H}(s))$;
     - ⋆ Variance $\sigma^2_{\widehat{H}}(t) = C_{\widehat{H}}(t,t)$ relies only on model parameters (independent with control functions);
     - ⋆ Control functions $c(\cdot)$ and $\kappa_i(\cdot)$ appear in the term $K(\cdot)$ only.
   - If $\mu_i = \mu$
     - ⋆ SVE degenerates to a time-varying Ornstein–Uhlenbeck (OU) process;
     - ⋆ SVE has an closed-form solution (so do $m_{\widehat{H}}(t)$ and $C_{\widehat{H}}(t,s)$).

$$\widehat{H}(t) = \frac{1}{R(t)} \left(\int_0^t \tilde{J}(u) d\mathcal{W}(u) + \int_0^t \tilde{R}(u) dK(u) + \int_0^t \tilde{K}(u) dR(u) \right).$$

Step III: Solve $c^*(t)$ and $\kappa_i^*(t)$ subject to Service-Level Constraints

- Road map: when n is large, at each time $t \in [0, T]$, we hope

$$\mathbb{P}(H_i^n(t) > w_i) = \mathbb{P}(\widehat{H}_i^n(t) > 0) \overbrace{\approx}^{n \text{ large}} \mathbb{P}(\widehat{H}_i(t) > 0)$$
$$\overbrace{=}^{\text{SSC}} \mathbb{P}(w_i(\widehat{H}(t) - \kappa_i(t)) > 0) \overbrace{=}^{\text{Gaussian dist.}} \mathbb{P}\left(\mathcal{N}\left(m_{\widehat{H}}(t), \sigma^2_{\widehat{H}}(t)\right) > \kappa_i(t)\right)$$
$$= \mathbb{P}\left(\mathcal{N}(0,1) > \frac{\kappa_i(t) - m_{\widehat{H}}(t)}{\sigma_{\widehat{H}}(t)}\right) \overbrace{\equiv}^{\text{want}} \alpha_i.$$

Recall that $m_{\widehat{H}}(t)$ is a function of $\kappa_i(t)$ and $c_i(t)$.

- Uniquely obtain the asymptotically "feasible" control functions:

$$c^*(t) = \sum_{i=1}^{K} \left(\eta_i(t)\kappa_i^*(t) - \int_0^t \eta_i(s) e^{\mu_i(s-t)} \left(\mu_i - h_{F_i}(w_i)\right) \kappa_i^*(s) ds\right),$$
$$\kappa_i^*(t) = z_{\alpha_i} \sigma_{\widehat{H}}(t), \qquad 1 \le i \le K, \quad 0 \le t \le T.$$

where $z_\alpha = \Phi^{-1}(1-\alpha)$, $\eta_i(t) \equiv w_i \lambda_i(t - w_i) F_i^c(w_i)$.

Step III: Solve $c^*(t)$ and $\kappa_i^*(t)$ subject to Service-Level Constraints

- Recall: delay-based **time-varying dynamic prioritization scheduling** rule:

$$i^* \in \arg\max_{1 \le i \le K} \left\{ \underbrace{\frac{H_i(t)}{w_i}}_{\textbf{normalized HWT}} + \underbrace{\frac{1}{\sqrt{\lambda^\star}} \kappa_i(t)}_{\textbf{second order}} \right\}.$$

  where

$$\kappa_i(t) = \underbrace{z_{\alpha_i}}_{\text{quantile}} \cdot \underbrace{\sigma_{\widehat{H}}(t)}_{\text{variability}}, \qquad 1 \le i \le K, \quad 0 \le t \le T.$$

- Some insights:
  - Sign of κ_i: when $\alpha_i \le 0.5$ (> 0.5), then $\kappa_i \ge 0$ (< 0)
    Adding (removing) some weight to make class i more (less) important;
  - Difference of κ_i: if $\alpha_1 < 0.5 < \alpha_2$, then $\kappa_1 > 0 > \kappa_2$, and $\kappa_1(t) - \kappa_2(t) = (z_{\alpha_1} - z_{\alpha_2})\sigma_{\hat{H}}(t)$;
    In a more (less) random system, need more (less) contrasting κ_i's;
  - TV-DPS is both time-dependent (accounting for time variability) and state-dependent (capturing stochasticity).

Step VI: Prove Asymptotic Optimality

The asymptotically "feasible" control functions:

$$c^*(t) = \sum_{i=1}^{K} \left(\eta_i(t)\kappa_i^*(t) - \int_0^t \eta_i(s) e^{\mu_i(s-t)} \left(\mu_i - h_{F_i}(w_i)\right) \kappa_i^*(s) ds \right), \qquad (1)$$

$$\kappa_i^*(t) = z_{\alpha_i} \sigma_{\widehat{H}}(t), \qquad 1 \le i \le K, \quad 0 \le t \le T. \qquad (2)$$

Theorem (Asymptotic service differentiation)

Under our staffing and scheduling rule with $c^*(\cdot)$ and $\kappa_i^*(\cdot)$ in (1) and (2),

(i) *Mean PWT and HWT are both asymptotically differentiated and stabilized:*

$$\mathbb{E}[W_i^n(t)] \to w_i \quad \text{and} \quad \mathbb{E}[H_i^n(t)] \to w_i \qquad \text{as} \quad n \to \infty, \qquad \text{for} \quad 0 < t \le T, \ 1 \le i \le K.$$

(ii) *TPoDs for PWT and HWT are both asymptotically differentiated and stabilized:*

$$\mathbb{P}(W_i^n(t) > w_i) \to \alpha_i \quad \text{and} \quad \mathbb{P}(H_i^n(t) > w_i) \to \alpha_i \qquad \text{as} \quad n \to \infty$$

for $0 < t \le T, \ 1 \le i \le K.$

Review of the Approach

$$s(t) = m(t) + (\lambda^\star)^{1/2} c(t)$$

$$i^* \in \arg\max_{1\le i\le K} \left\{ H_i(t)/w_i + \frac{1}{\sqrt{\lambda^\star}} \kappa_i(t) \right\}$$

$$\left(\widehat{H}_1^n, \dots, \widehat{H}_K^n\right) \Rightarrow \left(\widehat{H}_1, \dots, \widehat{H}_K\right)$$

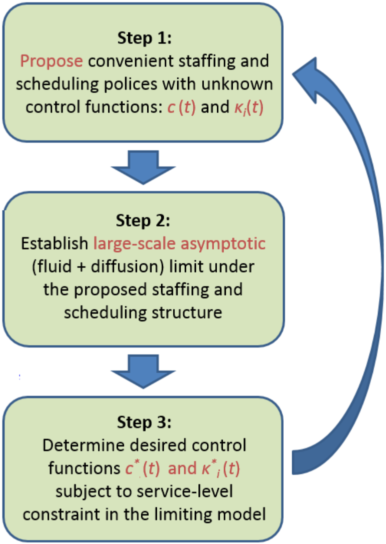
$$\widehat{H}_i(t) \equiv w_i(\widehat{H}(t) - \kappa_i(t)) \qquad \text{SSC}$$

$$\widehat{H}(t) = \int_0^t L(t,s)\widehat{H}(s)ds + \int_0^t J(t,s)dW(s) + K(t)$$

Analyzing $\widehat{H}(t)$ to obtain

$$\kappa_i^*(t) = z_{\alpha_i}\sigma_{\widehat{H}}(t)$$

$$c^*(t) = \cdots\cdot$$



Numerical Examples

Base Case - A Two-Class V Model

- Model parameters
  - Sinusoidal arrival rates $\lambda_i(t) = n\bar{\lambda}_i\,(1 + r_i \sin(\gamma_i t + \phi_i))$
    $\bar{\lambda}_1 = 1, \bar{\lambda}_2 = 1.5, r_1 = 0.2, r_2 = 0.3, \gamma_1 = \gamma_2 = 1, \phi_1 = 0, \phi_2 = -1$
  - Service rates $\mu_1 = \mu_2 = 1$ (later extend to class-dependent case)
  - Exponential abandonment times with rates $\theta_1 = 0.6, \theta_2 = 0.3$.
  - System scale: $n = 50$

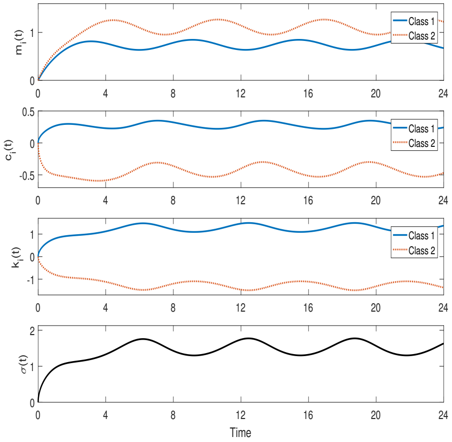
- QoS parameters
  - Delay targets $w_1 = 0.5, w_2 = 1$;
  - Probability targets $\alpha_1 = 0.2, \alpha_2 = 0.8$.

  Hope to achieve:

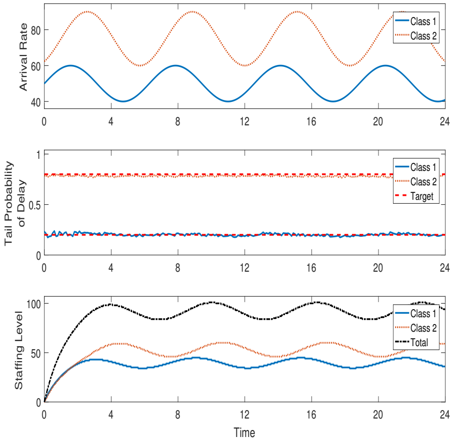
  $\mathbb{P}(W_1(t) > 0.5) \approx 20\%, \quad \mathbb{P}(W_2(t) > 1) \approx 80\%.$ Class-1 more important!

- Quality of service (QoS) increases as α_i and w_i decreases.
- Monte Carlo simulations of $\mathbb{P}(W_i(t) > w_i)$ with 5000 independent runs.

Base Case

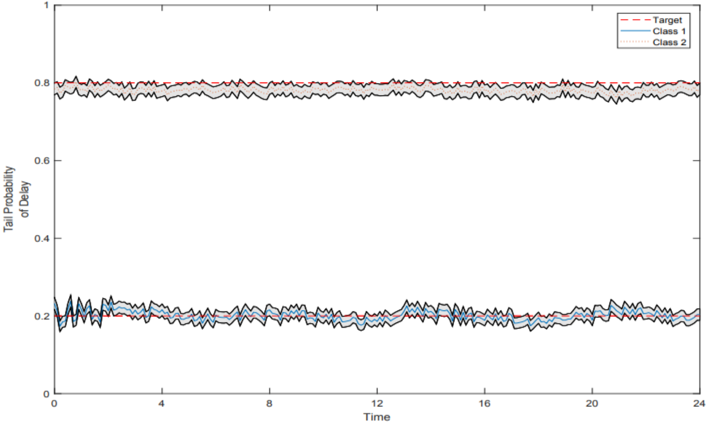


(a) Control functions



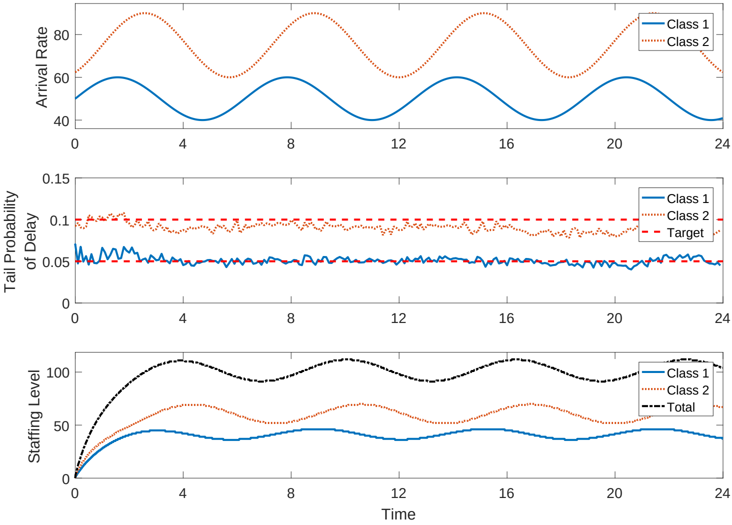
(b) Simulations

Base Case



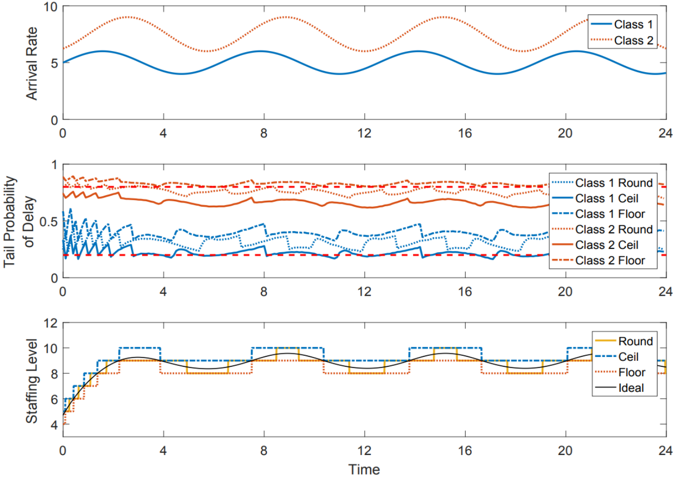
- Monte Carlo estimator: $\widehat{\mathbb{P}(W_i(t_j) > w_i)} \equiv \frac{1}{N}\sum_{k=1}^{N} \mathbf{1}(W_i^k(t_j) > w_i)$, $i = 1, 2$.
- $N = 5000$ samples, 99% confidence intervals, step size $\Delta t \equiv t_{j+1} - t_j = 0.01$.

High Quality of Service (Smaller α_i)



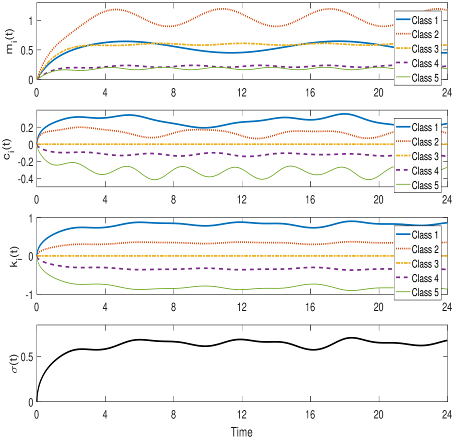
Good performance when $\alpha_i \leq 10\%$.

A Small System ($n = 5$)

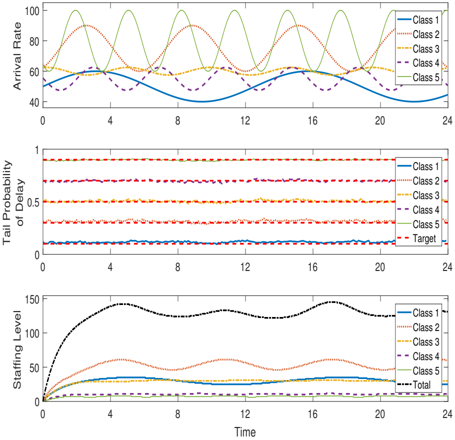


- When n is small, adding/removing a server causes bigger bumps;
- Error is attributed to discretization of staffing levels;
- Ok to apply large-scale results to a smaller scale system.

Five-Class Example



(c) Control solutions



(d) Simulations

Future Directions

- Differentiate PoD $\mathbb{P}(W_i(t) > 0) \approx \alpha_i$ (QED). Our scheduling rule

$$i^* \in \underset{1 \le i \le K}{\arg\max} \left\{ \frac{H_i^n(t)}{w_i} + \frac{1}{\sqrt{n}} \kappa_i(t) \right\} \quad \text{breaks down when } w_i = 0!$$

- Scheduling policies based on other states, e.g., queue length (information retrieval cost, implementation convenience, and server saving).
- Dependent arrival streams (modeling, simulation, and staffing).
- Nonexponential service distributions (non-Brownian driven limits).
- Multiple service pools (skill-based routing).
- Inflexible staffing functions (incorporating costs to change staffing levels).
- Customers switch classes (e.g., patient's health deteriorates/improves).

Queue-Length based Scheduling

- Implementation convenience (discrete counters)
- Information retrieval cost (dimensionality reduction)
- May require fewer servers??
  - QL provide more holistic understanding of current system state and is more forward looking (HoL delays only capture partial information and short-term dynamics); [1]
  - Under QL-based policy, system may be less variable, thus requiring less servers?

[1] Ibrahim, R. and W. Whitt. Real-Time Delay Estimation in Overloaded Multiserver Queues with Abandonment. *Management Science*, 55(10), 2009.

Queue-Length based Scheduling

- Same goal: $\mathbb{P}\left(W_i(t) > w_i\right) \approx \alpha_i$.
- A time-varying square-root staffing (TV-SRS) rule

$$s(t) = \underbrace{m(t)}_{\text{first order}} + \underbrace{\sqrt{\lambda^\star}c(t)}_{\text{second order}}, \qquad c(t) \text{ is TBD}$$

- A queue-length-based scheduling rule:
  Assigns the next available server to the HoL customer from class i^* satisfying

$$i^* \in \arg\max_{1\le i\le K} \left\{ \frac{Q_i^n(t)}{Q_\Sigma^n(t)} - \frac{q_i(t) + n^{-1/2}\hat{q}_i(t)}{q_\Sigma(t) + n^{-1/2}\hat{q}_\Sigma(t)} \right\}.$$

  - $Q_\Sigma(t) \equiv \sum_{i=1}^K Q_i(t)$, $Q_i(t)$ is the class-i waiting queue length.
  - $q_\Sigma(t) = \sum_{i=1}^K q_i(t)$, $q_i(t)$ is the class-i fluid queue length

$$q_i(t) = \int_{t-w_i}^{t} F_i^c(t-u)\lambda_i(u)\mathrm{d}u.$$

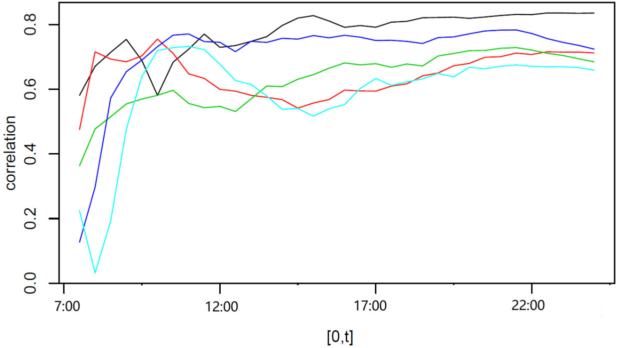
  - $\hat{q}_\Sigma(t) = \sum_{i=1}^K \hat{q}_i(t)$, $\hat{q}_i(t)$ is the second-order regulator (TBD).

Dependent Arrival Streams

Dependent Arrival Streams

- Data source: An American call center;
- Two customer types: VIP and regular;
- Time-dependent (positive) correlation for five weekdays

$$\rho_{1,2}(t) \equiv \frac{Cov(N_1(t), N_2(t))}{\sqrt{Var(N_1(t))Var(N_2(t))}} > 0.$$

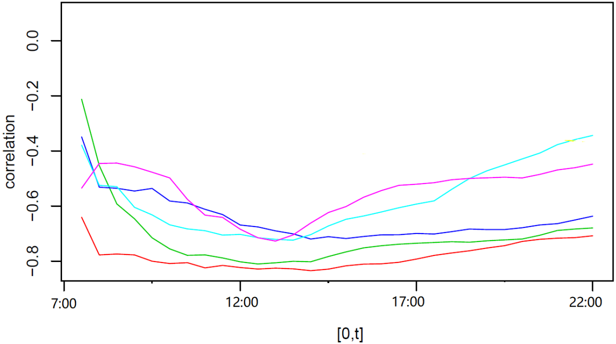


- Explanation: hidden (random) environmental process drives both classes.

Dependent Arrival Streams

- Data source: An Israeli call center;
- Two customer types: different service options;
- Time-dependent (negative) correlation on weekdays and weekends

$$\rho_{1,2}(t) \equiv \frac{Cov(N_1(t), N_2(t))}{\sqrt{Var(N_1(t))Var(N_2(t))}} < 0.$$



- Explanation: two classes share the same (finite) customer pool.

Dependent Arrival Streams

Interesting research questions:

- How to model dependent arrival process (each with time-varying rate)?
  - For example: given parameters $\lambda_i(t)$, $\rho_{i,j}(t)$, $1 \leq i, j \leq m$, how to properly define m-dim process $\{(N_1(t), \ldots, N_m(t)), 0 \leq t \leq T\}$?
- What are relevant statistical procedures to fit the model parameters from real data?
- How to effectively simulate such a model?
- What are the impacts of the dependent arrivals on
  - system performance (e.g., queue length, delay, etc.)
  - operational decisions (e.g., staffing and scheduling policies subject to service-level constraints).
    - ★ For positive/negative correlated arrival streams, should we add/remove some servers to meet a service target?
    - ★ If so, how many should we add and how does this dependent on $\rho_{i,j}$?

THANK YOU!

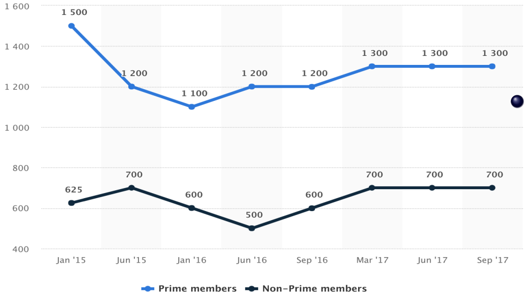
References

- Liu, Y. (2018). Staffing to stabilize the tail probability of delay in service systems with time-varying demand. Operations Research, 66(2), 514–534.
- Hovey, K., Liu, Y., & Sun, X. (2018). Staffing and Scheduling to Differentiate Service in Multiclass Time-Varying Service Systems. Submitted to Operations Research. https://yunanliu.wordpress.ncsu.edu/files/2019/01/SLED_OR-20180103.pdf
- Ding, Y., Park, E., Nagarajan, M., & Grafstein, E. (2018) Patient Prioritization in Emergency Department Triage Systems: An Empirical Study of Canadian Triage and Acuity Scale (CTAS). Manufacturing & Service Operations Management.
- Eick, S. G., Massey, W. A., & Whitt, W. (1993). The physics of the $M_t/G/\infty$ queue. Operations Research, 41(4), 731–742.
- Feldman, Z., Mandelbaum, A., Massey, W. A., & Whitt, W. (2008). Staffing of time-varying queues to achieve time-stable performance. Management Science, 54(2), 324–338.
- Gurvich, I., Armony, M., & Mandelbaum, A. (2008). Service-level differentiation in call centers with fully flexible servers. Management Science, 54(2), 279–294.

References

- Gurvich, I., & Whitt, W. (2010). Service-level differentiation in many-server service systems via queue-ratio routing. Operations research, 58(2), 316–328.
- Kim, J., Randhawa, R., Ward, A. R. (2018). Dynamic Scheduling in a Many-Server Multi-Class System: the Role of Customer Impatience in Large Systems. Manufacturing & Service Operations Management, 20(2), 285–301.
- Liu, Y., & Whitt, W. (2012). Stabilizing customer abandonment in many-server queues with time-varying arrivals. Operations research, 60(6), 1551–1564.
- Liu, Y., & Whitt, W. (2014). Many-server heavy-traffic limit for queues with time-varying parameters. The Annals of Applied Probability, 24(1), 378–421.
- Pender, J., & Massey, W. A. (2017). Approximating and stabilizing dynamic rate Jackson networks with abandonment. Probability in the Engineering and Informational Sciences, 31(1), 1–42.
- Soh, S. B., & Gurvich, I. (2016). Call center staffing: Service-level constraints and index priorities. Operations Research, 65(2), 537–555.

Another Motivating Example - Electronic Commerce



- Delivery guarantee
  - Prime member: within 24 hours
  - Regular member: within 4 days
- Non-stationary demand
  - How to determine the fleet size?
  - How to schedule shipment date?

Full Description of the Main Theorem

Suppose the system operates under the TV-SRS staffing and TV-DP scheduling rule. Then there is a joint convergence for the CLT-scaled processes:

$$\left(\widehat{H}_1^n, \ldots, \widehat{H}_K^n, \widehat{V}_1^n \ldots, \widehat{V}_K^n, \hat{X}_1^n, \ldots, \hat{X}_K^n, \hat{Q}_1^n, \ldots, \hat{Q}_K^n\right)$$
$$\Rightarrow \left(\widehat{H}_1, \ldots, \widehat{H}_K, \widehat{V}_1 \ldots, \widehat{V}_K, \hat{X}_1, \ldots, \hat{X}_K, \hat{Q}_1, \ldots, \hat{Q}_K\right) \quad \text{in} \quad \mathcal{D}^{4K} \quad \text{as} \quad n \to \infty,$$

where all limiting waiting-time processes can be expressed in terms of a one-dimensional process $\widehat{H}(\cdot)$:

$$\widehat{H}_i(t) \equiv w_i(\widehat{H}(t) - \kappa_i(t)), \qquad \widehat{V}_i(t) = w_i(\widehat{H}(t + w_i) - \kappa_i(t + w_i));$$

the process $\hat{H}$ uniquely solves the following *stochastic Volterra equation*

$$\widehat{H}(t) = \int_0^t L(t,s)\widehat{H}(s)ds + \int_0^t J(t,s)dW(s) + K(t),$$

where W is a standard Brownian motion,

$$L(t,s) \equiv \frac{\sum_{i=1}^K \eta_i(s) e^{\mu_i(s-t)} \left(\mu_i - h_{F_i}(w_i)\right)}{\eta(t)}, \quad J(t,s) \equiv \frac{\sqrt{\sum_{i=1}^K e^{2\mu_i(s-t)} (F_i^c(w_i)\lambda_i(s - w_i) + \mu_i m_i(s))}}{\eta(t)},$$
$$K(t) \equiv \frac{\sum_{i=1}^K \left(\eta_i(t)\kappa_i(t) - \int_0^t \eta_i(s) e^{\mu_i(s-t)} \left(\mu_i - h_{F_i}(w_i)\right) \kappa_i(s) ds\right) - c(t)}{\eta(t)}$$

for $\eta_i(t) \equiv w_i \lambda_i(t - w_i) F_i^c(w_i)$ and $\eta(t) \equiv \sum_{i \in \mathcal{I}} \eta_i(t)$.

Functional Weak Law of Large Numbers

The limit for each queue-length process can be decomposed into three terms:

$$\hat{Q}_i(t) \equiv \hat{Q}_{i,1}(t) + \hat{Q}_{i,2}(t) + \hat{Q}_{i,3}(t)$$
$$\hat{Q}_{i,1}(t) \equiv \int_{t-w_i}^{t} F_i^c(t-u)\sqrt{\lambda_i(u)}\mathrm{d}W_{\lambda_i}(u),$$
$$\hat{Q}_{i,2}(t) \equiv \int_{t-w_i}^{t} \sqrt{F_i^c(t-u)F_i(t-u)\lambda_i(u)}\mathrm{d}W_{\theta_i}(s),$$
$$\hat{Q}_{i,3}(t) \equiv \lambda_i(t-w_i)F_i^c(w_i)\widehat{H}_i(t),$$

for $W_{\lambda_i}, W_{\theta_i}, W_{\mu_i}$ being independent standard Brownian motions. Finally, the limits for number in system is given by $\hat{X}_i(t) = \hat{B}_i(t) + \hat{Q}_i(t)$.

As an immediate consequence of the FCLT result, we have

$$\left(\bar{B}_1^n, \ldots, \bar{B}_K^n, \bar{Q}_1^n, \ldots, \bar{Q}_K^n, \bar{X}_1^n, \ldots, \bar{X}_K^n, H_1^n, \ldots, H_K^n, V_1^n, \ldots, V_K^n\right)$$
$$\Rightarrow (m_1, \ldots, m_K, q_1, \ldots, q_K, x_1, \ldots, x_K, w_1\mathfrak{e}, \ldots, w_K\mathfrak{e}, w_1\mathfrak{e}, \ldots, w_K\mathfrak{e}) \quad \text{in} \quad \mathcal{D}^{5K}$$

as $n \to \infty$ where $\mathfrak{e}$ denotes constant function of one.

Computing $C(t, s)$

Algorithm:

(i) Pick an initial candidate $C^{(0)}(\cdot, \cdot)$;

(ii) In the k^{th} iteration, let $C^{(k+1)} = \Theta\left(C^{(i)}\right)$ with Θ given by

$$\Theta(C_{\widehat{H}})(t,s) = -\int_0^t \int_0^s L(t,u)L(s,v)C_{\widehat{H}}(u,v)dvdu + \int_0^t L(t,u)C_{\widehat{H}}(u,s)du + \int_0^s L(s,v)C_{\widehat{H}}(t,v)dv + \int_0^{s\wedge t} J(t,u)J(s,u)du.$$

Here $\Theta(\cdot)$ is a contraction operator.

(iii) If $\|C^{(k+1)} - C^{(k)}\|_T < \epsilon$, stop; otherwise, $k = k + 1$ and go back to step (ii).

According to the Banach contraction theorem, this algorithm should converge exponentially fast. Finally, we take $\text{Var}(\hat{H}(t)) = C(t,t)$, for $0 \leq t \leq T$.

Part I - Single Class Model

$M_t/M/s_t + GI$

- Nonhomogenous Process arrivals (easily extendable)
- I.I.D. exponential service times with rate μ (great difficulty arises when extended to general services)
- Time-varying staffing level (TBD)
- I.I.D. abandonment times $\sim F(x) \equiv \mathbb{P}(A \leq x)$ (the $+GI$)
- First-Come First-Served
- Unlimited waiting capacity

Performance functions

- $Q(t)$ and $B(t)$: number in queue and in service at time t
- $X(t) \equiv Q(t) + B(t)$: total number in system at time t
- $V(t)$: potential waiting time at time t

Staffing to Reduce Excessive Delay

- Objective: $\mathbb{P}(V(t) > w) \approx \alpha \in (0, 1)$
- Key idea: $V(t)$ is approximately normal for λ and s large (L&W14)
- Propose staffing:

$$s(t) = \lceil m(t) + \tilde{c}(t) \rceil \tag{3}$$

- Detailed Formula:

$$m(t) = F^c(w) \int_0^t e^{-\mu(t-u)} \lambda(u - w) \mathrm{d}u \quad \text{(offered-load process)} \tag{4}$$

$$c(t) = z_\alpha e^{-\mu t} \left(Z(t) - (\mu - h_F(w)) \int_0^t Z(s) ds \right) \tag{5}$$

$$\text{for} \quad Z(t) \equiv e^{(\mu - h_F(w))t} \sqrt{\int_0^t e^{2h_F(w)} \left(F^c(w) \lambda(u - w) + \mu m(u) \right)}, \tag{6}$$

$z_\alpha = \Phi^{-1}(1 - \alpha)$ and $h_F(x) \equiv f(x)/F^c(x)$.

- Formula (4) was derived by L&W12 and (5) - (6) came from L18.

Heuristic Derivation of $m(\cdot)$

1. Consider an $M_t/GI/\infty$ model with arrival from time zero. Then number in system $Q(t) \sim$ Poisson r.v. with mean (EMW93)

$$m(t) \equiv \mathbb{E}[Q(t)] = \int_0^t \lambda(u) G^c(t-u)\mathrm{d}u = \int_0^t \lambda(t-s) G^c(s)\mathrm{d}s.$$

   With exponential services we have $G^c(x) = e^{-\mu x}$, and so

$$m(t) = \int_0^t \lambda(u) e^{-\mu(t-u)}\mathrm{d}u. \tag{7}$$

2. If the mean waiting time is stabilized at the target w, then on average a customer (if not abandon) will wait w time units before entering service.
3. Hence the "effective arrival" rate $\tilde{\lambda}(t) \equiv \lambda(t-w)F^c(w)$. Replacing $\lambda(t)$ in (7) with $\tilde{\lambda}(t)$ yields (4), as desired!
4. In summary, the offered load $m(t)$ is the mean number of busy servers needed to serve all customers who are willing to wait (hence excluding an acceptable faction of customer abandoned).

A Five-Class V Model: Parameters and QoS Targets

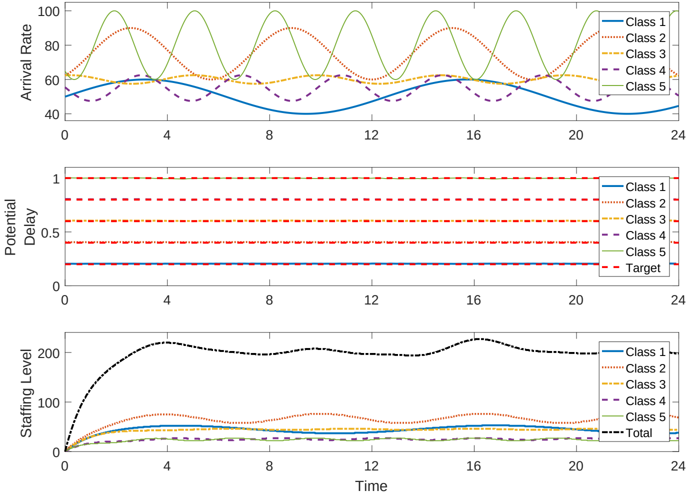
- Sinusoidal arrival rates $\lambda_i(t) = n\bar{\lambda}_i \left(1 + r_i \sin(\gamma_i t + \phi_i)\right)$;
- Exponential service times;
- Exponential abandonment times;
- Scale $n = 50$.

	Arrival Parameters				Abandonment rates	Service rates	Service levels	
Class	$\bar{\lambda}_i$	r_i	γ_i	ϕ_i	θ_i	μ_i	w_i	α_i
1	1.0	0.20	1	0	0.6	1	0.2	0.1
2	1.5	0.30	1	-1	0.3	1	0.4	0.3
3	1.2	0.05	1	1	0.5	1	0.6	0.5
4	1.1	0.15	1	-2	1.0	1	0.8	0.7
5	1.6	0.40	1	2	1.2	1	1.0	0.9

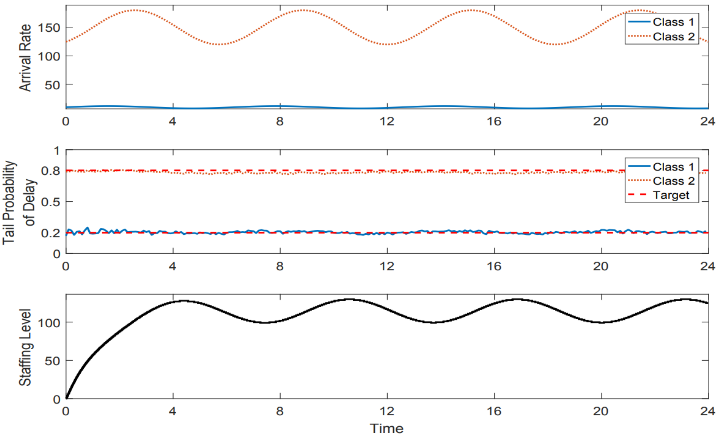
The priority decreases in i, $1 \leq i \leq 5$.

Five-Class Example

- Goal: Stabilizing mean waiting time $\mathbb{E}[W_i(t)] = w_i$, $(w_1, w_2, w_3, w_4, w_5) = (0.2, 0.4, 0.6, 0.8, 1)$.
- Apply our staffing and scheduling rule with $\alpha_i = 1/2$, $1 \leq i \leq 5$.

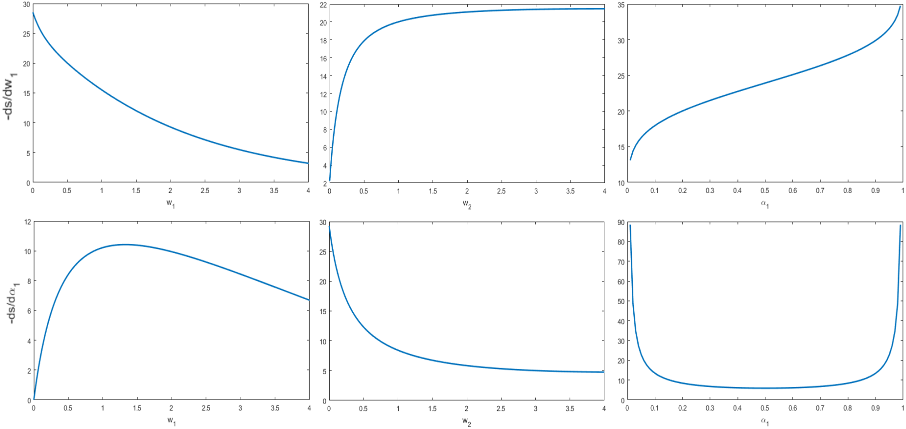


Mixed Arrival



Base Case: Constant Arrival Rate

$\lambda_1 = n = 50$, $\lambda_2 = 1.5n = 75$, $(w_1, w_2, \alpha_1, \alpha_2) = (0.5, 1, 0.2, 0.8)$.

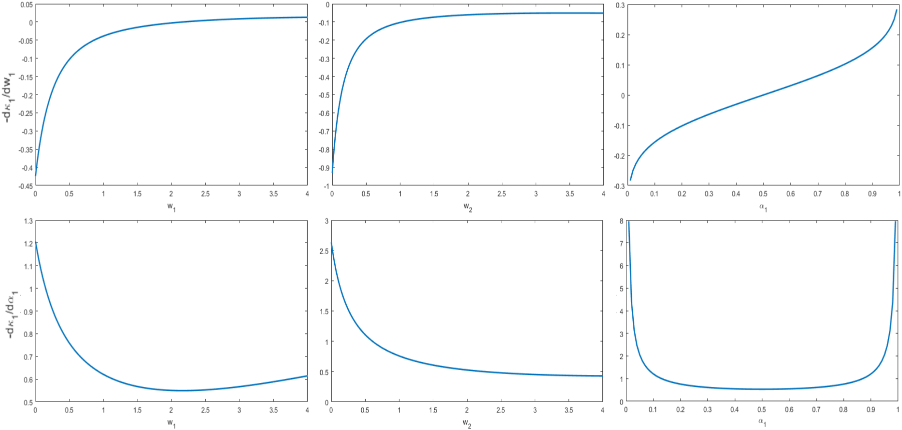


For example, reducing w_1 from 0.5 to 0.4, need to add

$$\Delta s \approx \left(-\frac{\partial s}{\partial w_1}\right) \times 0.1 \approx 20 \times 0.1 = 2.$$

Base Case: Constant Arrival Rate

$\lambda_1 = n = 50$, $\lambda_2 = 1.5n = 75$, $(w_1, w_2, \alpha_1, \alpha_2) = (0.5, 1, 0.2, 0.8)$.

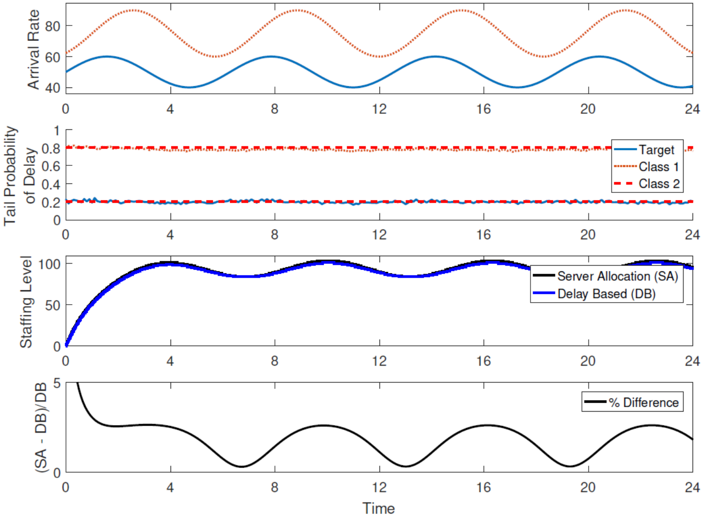


For example, reducing α_1 from 0.2 to 0.15, need to increase κ_1 by

$$\Delta\kappa_1 \approx \left(-\frac{\partial\kappa_1}{\partial\alpha_1}\right) \times 0.05 \approx 0.75 \times 0.05 = 0.0375.$$

Server-Based Scheduling

- Compute $s_1(t)$ and $s_2(t)$ using single-class formulas;
- Schedule customers based on server ratio $s_1(t)/s_2(t)$.



Special Case: Class-Independent Service $\mu_i = \mu$ and Constant Arrival $\lambda_i(t) = \lambda_i$

- Staffing

$$m_i(t) \sim m_i \equiv \frac{\lambda_i F_i^c(w_i)}{\mu}, \qquad c(t) \sim c \equiv \sum_{i=1}^{K} \frac{w_i \lambda_i f_i(w_i)}{\mu} \kappa_i,$$

- Scheduling

$$\kappa_i(t) \sim \kappa_i \equiv z_{\alpha_i} \cdot \sqrt{\frac{\sum_{j=1}^{K} \lambda_j F_j^c(w_j)}{\left(\sum_{j=1}^{K} \lambda_j f_j(w_j) w_j\right)\left(\sum_{j=1}^{K} \lambda_j F_j^c(w_j) w_j\right)}}.$$

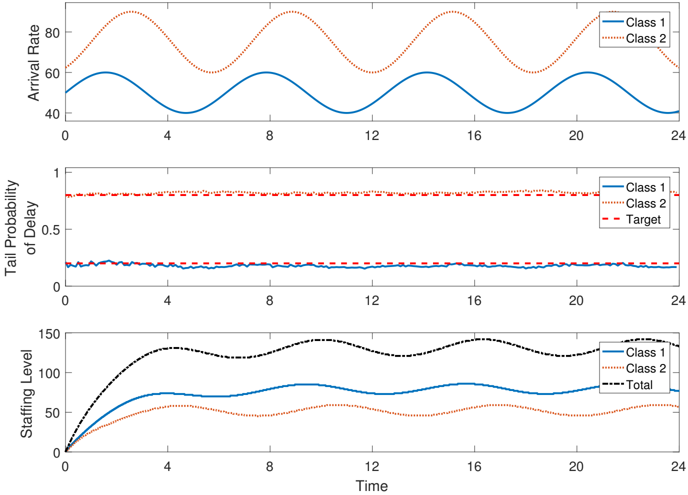
These formulas can be used to estimate

- the required server capacity and scheduling threshold;
- the *marginal price of staffing and scheduling* (MPSS):

  To improve the service to the next level ($w_i \to w_i - \Delta w_i$ or $\alpha_i \to \alpha_i - \Delta\alpha_i$), how many extra servers Δs are needed and how to adjust the scheduling threshold κ_i?

$$\Delta s = \left(-\frac{\partial s(\mathbf{w}, \boldsymbol{\alpha})}{\partial w_i}\right) \Delta w_i, \qquad \Delta \kappa_i = \left(-\frac{\partial \kappa_i(\mathbf{w}, \boldsymbol{\alpha})}{\partial \alpha_i}\right) \Delta \alpha_i,$$

Class-Dependent Service Rates ($\mu_1 = 0.5, \mu_2 = 1$)



- $\sigma^2_{\widehat{H}}(t)$ is numerically computed using our algorithm.
- It takes 42 iterations to converge with $\epsilon = 10^{-6}$.

Many-Server Limit of Waiting Times and SSC

Assumption: Exponential abandonment-time distributions with rate θ_i.

Under SRS staffing and QL-based scheduling:

$$\left(\widehat{H}_1^n, \ldots, \widehat{H}_K^n, \hat{X}_1^n \ldots, \hat{X}_K^n\right) \Rightarrow \left(\widehat{H}_1, \ldots, \widehat{H}_K, \hat{X}_1 \ldots, \hat{X}_K\right) \quad \text{in} \quad \mathcal{D}^{2K} \quad \text{as} \quad n \to \infty,$$

with total queue length $\hat{\mathbf{X}} = \left(\hat{X}_1, \ldots, \hat{X}_K\right)$ solving a K-dim time-varying OU SDE

$$\mathrm{d}\hat{\mathbf{X}}(t) = \mathbf{A}(t)\hat{\mathbf{X}}(t)\mathrm{d}t + \mathbf{B}(t)\mathrm{d}\boldsymbol{\mathcal{W}}(t) + \mathbf{C}(t)dt, \qquad \hat{X}_\Sigma(t) = \sum_{i=1}^{K} \hat{X}_i(t),$$

$$\widehat{H}_i(t) = \frac{e^{\theta_i w_i}}{\lambda_i(t)}\left[\left(p_i(t)(\hat{X}_\Sigma(t) - c(t) - \hat{q}_\Sigma(t)) + \hat{q}_i(t)\right) - \int_{t-w_i}^{t} e^{-\theta_i(t-u)}\sqrt{\lambda_i(t-u)}\mathrm{d}W_{\lambda_i}(u) - \int_{t-w_i}^{t} \sqrt{e^{-\theta_i(t-u)}(1-e^{-\theta_i(t-u)})}\sqrt{\lambda_i(t-u)}\mathrm{d}W_{\theta_i}(u)\right].$$

Many-Server Limit of Waiting Times and SSC

$$\mathrm{d}\hat{\mathbf{X}}(t) = \mathbf{A}(t)\hat{\mathbf{X}}(t)\mathrm{d}t + \mathbf{B}(t)\mathrm{d}\boldsymbol{\mathcal{W}}(t) + \mathbf{C}(t)dt,$$

where $\hat{\mathbf{X}}(t) \equiv [\hat{X}_1(t), \ldots, \hat{X}_K(t)]^T$,

$$\mathbf{A}(t) \equiv -\begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_K \end{bmatrix} + \begin{bmatrix} (\mu_1-\theta_1)p_1(t) & \cdots & (\mu_1-\theta_1)p_1(t) \\ \vdots & \vdots & \vdots \\ (\mu_K-\theta_K)p_K(t) & \cdots & (\mu_K-\theta_K)p_K(t) \end{bmatrix},$$

$$\mathbf{B}(t) \equiv \begin{bmatrix} \sqrt{\lambda_1(t)+\mu_1 m_1(u)+\theta_1 q_1(u)} & & \\ & \ddots & \\ & & \sqrt{\lambda_m(t)+\mu_K m_K(u)+\theta_K q_K(u)} \end{bmatrix},$$

$$\mathbf{C}(t) \equiv -(c(t)+\hat{q}_\Sigma(t)) \begin{bmatrix} (\mu_1-\theta_1)p_1(t) \\ \vdots \\ (\mu_K-\theta_K)p_K(t) \end{bmatrix} + \begin{bmatrix} (\mu_1-\theta_1)\hat{q}_1(t) \\ \vdots \\ (\mu_K-\theta_K)\hat{q}_K(t) \end{bmatrix},$$

$$p_i(t) \equiv \frac{q_i(t)}{q_\Sigma(t)},$$

and $\boldsymbol{\mathcal{W}}(t) \equiv [\mathcal{W}_1(t), \ldots, \mathcal{W}_K(t)]^T$ is a K-dim independent BM.

Many-Server Limit of Waiting Times and SSC

Next, matching the TPoD to desired probability target

$$\mathbb{P}(H_i^n(t) > w_i) = \mathbb{P}(\widehat{H}_i^n(t) > 0) \approx \mathbb{P}(\widehat{H}_i(t) > 0) = \cdots = \alpha_i,$$

yields "unique" control functions

- scheduling regulators

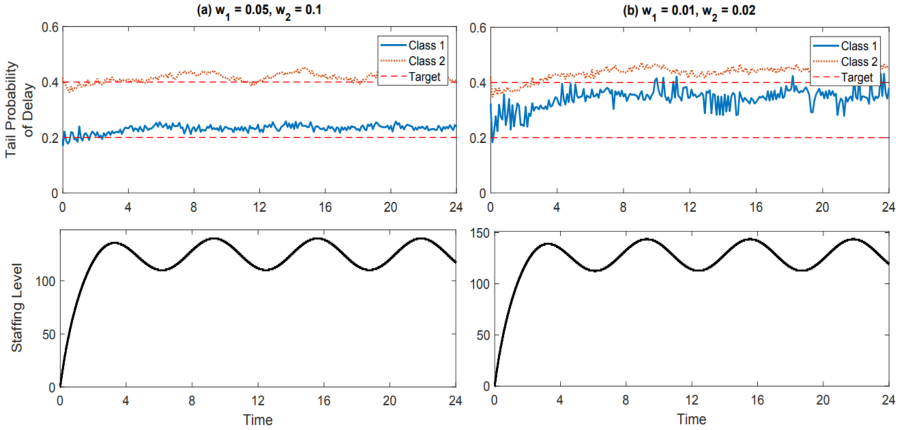
$$\hat{q}_i(t) = z_{\alpha_i}\ \tilde{\sigma}(t), \qquad 1 \le i \le K.$$

- our desired second-order staffing function c^* solves a *fixed-point equation*

$$c = \Psi(c), \quad \text{where} \quad \Psi(c)(t) \equiv \beta(t) + \mathbf{e} \int_0^t e^{\int_u^t \mathbf{A}(v)\mathrm{d}v} \mathbf{C}(u)\mathrm{d}u.$$

Remark: When $\mu_i = \mu$, $c(t)$ has a closed-form solution.

Very High Quality of Service (Smaller w_i)



- If $w_i \approx 0$, TPoD degenerates to probability of delay (PoD) $\mathbb{P}(W_i(t) > 0)$.
- Our method will eventually break down...

Fixed Staffing Intervals

Average staffing level (ASL) and *max staffing level* (MSL)

$$s^{\mathrm{ASL}}(t) \equiv \sum_{i=1}^{\lceil T/\Delta_s \rceil} \bar{s}_i \mathbf{1}_{\{t \in [(i-1)\Delta_s, i\Delta_s\}}, \qquad \bar{s}_i \equiv \frac{1}{\Delta_s} \int_{(i-1)\Delta_s}^{i\Delta_s \wedge T} s(u)\mathrm{d}u,$$

$$s^{\mathrm{MSL}}(t) \equiv \sum_{i=1}^{\lceil T/\Delta_s \rceil} s_i^{\uparrow} \mathbf{1}_{\{t \in [(i-1)\Delta_s, i\Delta_s\}}, \qquad s_i^{\uparrow} \equiv \sup_{(i-1)\Delta_s \le u \le i\Delta_s \wedge T} s(u),$$