Online Supplement

to

"Modeling and Simulation of Nonstationary and Non-Poisson Processes"

by

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This Online Supplement includes additional materials supplementing the main paper. In Section S1 we prove Theorems 1 and 2, the main asymptotic properties of CIATA-Ph. In Section S2 we give the proofs of Propositions 1–2. In Section 4.5 we compare the performance of CIATA-Ph with that of the algorithms of Gerhardt and Nelson (2009). In Section S3 we present additional results supplementing the numerical experiments in the main paper.

S1. Proofs of the Key Properties of CIATA-Ph

Throughout this section, $\mathbb{R}_+$ denotes the nonnegative real numbers; and for any function $\vartheta(t)$ defined on $[0,S]$, summations of the form $\sum_{n=1}^{\widetilde{N}_Q(S)} \vartheta(\widetilde{S}_n)$ are taken to be identically 0 when $\widetilde{N}_Q(S)=0$. First we establish two auxiliary results that are used in the proof of Theorem 1.

LEMMA 1. *If Q is sufficiently large, then $0<\widetilde{\lambda}_Q(t)\le\lambda^*$ and $0\le\widetilde{\mu}_Q(t)\le\lambda^* t$ for $t\in[0,S]$; moreover $\widetilde{N}_Q(t)=N^\circ[\widetilde{\mu}_Q(t)]<\infty$ almost surely and $\mathrm{E}[\widetilde{N}_Q(t)]=\widetilde{\mu}_Q(t)$ for each $t\in[0,S]$.*

Proof of Lemma 1. Assumption 1, the constructed properties (4) and (v) of $\widetilde{\lambda}_Q(t)$ on $[0,S]$ as detailed in §3.2, and the definition (7) of $\widetilde{\mu}_Q(t)$ imply there is a $Q^*\ge 1$ such that when $Q\ge Q^*$ and $t\in[0,S]$, we have $0<\widetilde{\lambda}_Q(t)\le\lambda^*<\infty$ and $0\le\widetilde{\mu}_Q(t)\le\lambda^* t<\infty$. Since $\{N^\circ(u):u\ge 0\}$ is an ERP with $\mathrm{E}[X_2^\circ]=1$, Theorem 3.5.2 of Ross (1996) ensures that $\mathrm{E}[N^\circ(u)]=u$ for $u\ge 0$. Thus for each $t\in[0,S]$ and $Q\ge Q^*$, the event $\{N^\circ[\widetilde{\mu}_Q(t)]<\infty$ and $\widetilde{N}_Q(t)=N^\circ[\widetilde{\mu}_Q(t)]\}$ has probability 1 so that $\mathrm{E}[\widetilde{N}_Q(t)]=\mathrm{E}\{N^\circ[\widetilde{\mu}_Q(t)]\}=\widetilde{\mu}_Q(t)$. □

Lemma 2 below generalizes Proposition 9.1.14 in Çinlar (1975), which applies to an ordinary renewal process over an infinite time horizon such that $G(0)<1$ and the initial renewal epoch is taken to occur at time 0. Adapted to CIATA-Ph, Lemma 2 applies to the finite-horizon NNPP $\{\widetilde{N}_Q(t):t\in[0,S]\}$ obtained by inversion of the finite-horizon ERP $\{N^\circ(u):u\in[0,\widetilde{\mu}_Q(S)]\}$ for which $G(x)$ is continuous on $\mathbb{R}_+$, $G(0)=0$, and $G(x)<1$ for $x>0$. Hence with complementary positive probabilities, the CIATA-Ph–generated NNPP will have either (i) no arrivals in $[0,S]$; or (ii) the initial arrival epoch $\widetilde{S}_1\in(0,S]$. Throughout the rest of this section, we let Q^* denote the smallest value of Q for which the conclusions of Lemma 1 hold.

LEMMA 2. *If $f(t)$ is a nonnegative, bounded, measurable function defined on $\mathbb{R}_+$ that vanishes outside a finite interval, then for sufficiently large values of Q we have*

$$\mathrm{E}\left[\sum_{n=1}^{\widetilde{N}_Q(S)} f(\widetilde{S}_n)\right]=\int_0^S f(t)\widetilde{\lambda}_Q(t)\,\mathrm{d}t\,. \tag{S.1}$$

Proof of Lemma 2. Select any $Q \geq Q^*$. Let $[t_*, t^*]$ denote a finite interval outside of which $f(t)$ vanishes, and let f^* denote a finite upper bound for $f(t)$. To establish Equation (S.1) for $f(t)$, we start with $\mathbb{I}_{[a,\,b]}(t)$, the indicator function for an arbitrary interval $[a,b]$, where $0 \leq a \leq b < \infty$. Each of the ERP's renewal epochs $\{S_n^\circ : n = 1, \ldots, N^\circ[\widetilde{\mu}_Q(S)]\}$ is a continuous random variable with a probability density function (p.d.f.). By Assumption 1 and by the constructed properties of $\widetilde{\lambda}_Q(t)$ and $\widetilde{\mu}_Q(t)$, we see that $\widetilde{\lambda}_Q(t)$ is continuous almost everywhere on $[0,S]$ and $\widetilde{\mu}_Q(t)$ is piecewise linear and strictly increasing on $[0,S]$; therefore we can apply the general change-of-variable theorem for Riemann integration (Kestelman 1961) and Theorem 10.33 of Rudin (1964) to show that each of the NNPP's arrival epochs $\{\widetilde{S}_n : n = 1, \ldots, \widetilde{N}_Q(S)\}$ is also a continuous random variable with a p.d.f., and we have

$$\begin{aligned} \mathrm{E}\Bigg\{ \sum_{n=1}^{\widetilde{N}_Q(S)} \mathbb{I}_{[a,\,b]}(\widetilde{S}_n) \Bigg\} &= \mathrm{E}\Bigg\{ \sum_{n=1}^{\widetilde{N}_Q(S)} \mathbb{I}_{(a,\,b]}(\widetilde{S}_n) \Bigg\} = \mathrm{E}\big[\widetilde{N}_Q(\min\{b,S\}) - \widetilde{N}_Q(\min\{a,S\})\big] \\ &= \int_0^S \mathbb{I}_{(a,\,b]}(t)\widetilde{\lambda}_Q(t)\,\mathrm{d}t = \int_0^S \mathbb{I}_{[a,\,b]}(t)\widetilde{\lambda}_Q(t)\,\mathrm{d}t\,, \end{aligned} \tag{S.2}$$

where the next-to-last equality in Equation (S.2) follows from Lemma 1. A similar argument establishes a version of Equation (S.2) in which the function $\mathbb{I}_{[a,\,b]}(t)$ is replaced by $\mathbb{I}_{(a,\,b)}(t)$. If $\psi(t)$ is a step function on $\mathbb{R}_+$, then Lemma 1.1.3 of Asplund and Bungart (1966) ensures that $\psi(t)$ is a finite linear combination of indicator functions for mutually disjoint bounded intervals each of which is contained in $\mathbb{R}_+$ and is either open or closed. Lemma 1 and Equation (S.2) imply that

$$\mathrm{E}\Bigg[\sum_{n=1}^{\widetilde{N}_Q(S)} \psi(\widetilde{S}_n) \Bigg] = \int_0^S \psi(t)\widetilde{\lambda}_Q(t)\,\mathrm{d}t\,. \tag{S.3}$$

Since the bounded measurable function $f(t)$ vanishes outside the finite interval $[t_*, t^*]$, Theorem 4 in §4.2 of Royden and Fitzpatrick (2010) ensures that $f(t)$ is integrable; and by Proposition 2.1.8 of Asplund and Bungart (1966), there is a sequence of step functions $\{\varphi_k(t) : k \geq 1\}$ on $\mathbb{R}_+$ such that $|\varphi_k(t)| \leq f^*\mathbb{I}_{[t_*,\,t^*]}(t)$ for all $t \geq 0$ and $k \geq 1$, $\lim_{k\to\infty}\varphi_k(t) = f(t)$ for almost all $t \geq 0$, and $\lim_{k\to\infty}\int_0^\infty \varphi_k(t)\,\mathrm{d}t = \int_0^\infty f(t)\,\mathrm{d}t$. Lemma 1 ensures that $\big|\varphi_k(t)\widetilde{\lambda}_Q(t)\big| \leq f^*\lambda^*$ for all $t \in [0,S]$ and $k \geq 1$; and since $\lim_{k\to\infty}\varphi_k(t)\widetilde{\lambda}_Q(t) = f(t)\widetilde{\lambda}_Q(t)$ for almost all $t \geq 0$, it follows from the dominated convergence theorem (Billingsley 1995, Theorem 16.4) that

$$\lim_{k\to\infty}\int_0^S \varphi_k(t)\widetilde{\lambda}_Q(t)\,\mathrm{d}t = \int_0^S f(t)\widetilde{\lambda}_Q(t)\,\mathrm{d}t\,. \tag{S.4}$$

We complete the proof by another application of the convergence property $\lim_{k\to\infty}\varphi_k(t) = f(t)$ for almost all $t \geq 0$. Let $\mathscr{D} \subset \mathbb{R}_+$ denote a set of measure zero such that $\lim_{k\to\infty}\varphi_k(t) = f(t)$ for all $t \in \mathbb{R}_+ \setminus \mathscr{D}$. For $n \geq 1$, let $h_n(t)$ denote the p.d.f. of $\widetilde{S}_n$ on $\mathbb{R}_+$. Proposition 9 in Chapter 4 of Royden and Fitzpatrick (2010) ensures that $\Pr\{\widetilde{S}_n \in \mathscr{D}\} = \int_{\mathscr{D}} h_n(t)\,\mathrm{d}t = 0$ for $n \geq 1$. Therefore Theorem 2.1 of Billingsley (1995) ensures that with probability 1, we have $\widetilde{S}_n \in \mathbb{R}_+ \setminus \mathscr{D}$ for $n \geq 1$ and

$$\lim_{k\to\infty}\varphi_k(\widetilde{S}_n) = f(\widetilde{S}_n) \;\text{ for }\; n \geq 1 \;\text{ almost surely}\,. \tag{S.5}$$

Lemma 1 and Equation (S.5) imply that

$$\lim_{k\to\infty}\sum_{n=1}^{\widetilde{N}_Q(S)}\varphi_k(\widetilde{S}_n)=\sum_{n=1}^{\widetilde{N}_Q(S)}\Big[\lim_{k\to\infty}\varphi_k(\widetilde{S}_n)\Big]=\sum_{n=1}^{\widetilde{N}_Q(S)}f(\widetilde{S}_n)\text{ almost surely}. \tag{S.6}$$

Next we apply Equation (S.3) with $\psi(t)$ replaced successively by $\varphi_k(t)$ for $k\geq 1$; and we combine the resulting equations with the inequalities $\left|\sum_{n=1}^{\widetilde{N}_Q(S)}\varphi_k(\widetilde{S}_n)\right|\leq f^*\widetilde{N}_Q(S)$ for $k\geq 1$ and the convergence property (S.6) so that by applying the dominated convergence theorem to the left-hand side of Equation (S.3), we have

$$\lim_{k\to\infty}\int_0^S\varphi_k(t)\widetilde{\lambda}_Q(t)\,\mathrm{d}t=\mathrm{E}\Bigg[\sum_{n=1}^{\widetilde{N}_Q(S)}f(\widetilde{S}_n)\Bigg]. \tag{S.7}$$

Combining Equations (S.4) and (S.7) yields the conclusion of Lemma 2. □

Proof of Theorem 1. Select $Q\geq Q^*$ and $t\in[0,S]$ arbitrarily. We establish a series representation for $\mathrm{E}[N_Q(t)]$ in terms of the majorizing arrival epochs $\big\{\widetilde{S}_n:n\geq 0\big\}$ and their associated acceptance probabilities; and finally we apply Lemma 2 to complete the proof. We define the binary sequence $\{B_n:n\geq 0\}$ such that $B_0\equiv 0$; and for $n\geq 1$, we set $B_n=1$ if the nth majorizing arrival epoch $\widetilde{S}_n$ is accepted independently, and we set $B_n=0$ otherwise. Then $N_Q(t)=\sum_{n=0}^{\widetilde{N}_Q(S)}\mathbb{I}_{[0,\,t]}(\widetilde{S}_n)B_n$; and conditioning on $\widetilde{N}_Q(S)$, we have

$$\begin{aligned}
\mathrm{E}[N_Q(t)]&=\sum_{m=0}^{\infty}\Pr\{\widetilde{N}_Q(S)=m\}\mathrm{E}\Bigg[\sum_{n=0}^{\widetilde{N}_Q(S)}\mathbb{I}_{[0,\,t]}(\widetilde{S}_n)B_n\,\Bigg|\,\widetilde{N}_Q(S)=m\Bigg]\\
&=\sum_{m=0}^{\infty}\Pr\{\widetilde{N}_Q(S)=m\}\left\{\sum_{n=0}^{m}\mathrm{E}\big[\mathbb{I}_{[0,\,t]}(\widetilde{S}_n)B_n\,\big|\,\widetilde{N}_Q(S)=m\big]\right\}\\
&=\sum_{m=0}^{\infty}\Pr\{\widetilde{N}_Q(S)=m\}\left(\sum_{n=0}^{m}\mathrm{E}\Big\{\mathrm{E}\big[\mathbb{I}_{[0,\,t]}(\widetilde{S}_n)B_n\,\big|\,\widetilde{S}_n,\widetilde{N}_Q(S)=m\big]\,\Big|\,\widetilde{N}_Q(S)=m\Big\}\right)\\
&=\sum_{m=0}^{\infty}\Pr\{\widetilde{N}_Q(S)=m\}\left(\sum_{n=0}^{m}\mathrm{E}\Big\{\mathbb{I}_{[0,\,t]}(\widetilde{S}_n)\mathrm{E}\big[B_n\,\big|\,\widetilde{S}_n,\widetilde{N}_Q(S)=m\big]\,\Big|\,\widetilde{N}_Q(S)=m\Big\}\right).
\end{aligned} \tag{S.8}$$

For $n=0$ in Equation (S.8), we have $\mathrm{E}\big[B_0\,\big|\,\widetilde{S}_0,\widetilde{N}_Q(S)=m\big]=0$ for each $m\geq 0$ because $B_0\equiv 0$. Given $\widetilde{S}_n$ and $\widetilde{N}_Q(S)=m$ for $1\leq n\leq m$, we have $\widetilde{S}_n\leq S$ and the arrival epoch $\widetilde{S}_n$ is accepted independently with probability $\mathrm{E}\big[B_n\,\big|\,\widetilde{S}_n,\widetilde{N}_Q(S)=m\big]=\lambda(\widetilde{S}_n)\big/\widetilde{\lambda}_Q(\widetilde{S}_n)$. Inserting these results into Equation (S.8), we have

$$\begin{aligned}
\mathrm{E}[N_Q(t)]&=\sum_{m=0}^{\infty}\Pr\{\widetilde{N}_Q(S)=m\}\left(\sum_{n=1}^{m}\mathrm{E}\big[\mathbb{I}_{[0,\,t]}(\widetilde{S}_n)\lambda(\widetilde{S}_n)\big/\widetilde{\lambda}_Q(\widetilde{S}_n)\,\big|\,\widetilde{N}_Q(S)=m\big]\right)\\
&=\mathrm{E}\Bigg[\sum_{n=1}^{\widetilde{N}_Q(S)}\mathbb{I}_{[0,\,t]}(\widetilde{S}_n)\lambda(\widetilde{S}_n)\big/\widetilde{\lambda}_Q(\widetilde{S}_n)\Bigg].
\end{aligned} \tag{S.9}$$

To show that $\mathrm{E}[N_Q(t)]=\mu(t)$, we verify that the function $f_t(y)\equiv\mathbb{I}_{[0,\,t]}(y)\lambda(y)\big/\widetilde{\lambda}_Q(y)$ for $y\in[0,S]$ and $f_t(y)\equiv 0$ for $y>S$ satisfies the assumptions of Lemma 2. Because $0\leq\lambda(y)\leq\widetilde{\lambda}_Q(y)$ and $\widetilde{\lambda}_Q(y)>0$ on $[0,S]$, we see that $0\leq f_t(y)\leq 1$ on $[0,S]$. Moreover, because $\mathbb{I}_{[0,\,t]}(y)$, $\lambda(y)$, and $\widetilde{\lambda}_Q(y)$ are all measurable functions on $[0,S]$ and $\widetilde{\lambda}_Q(y) > 0$ on $[0,S]$, it follows that the function $f_t(y)$ is measurable on $[0,S]$; see the Corollary to Theorem 10 on p. 288 of Kolmogorov and Fomin (1970). Therefore we have

$$\int_0^S f_t(y)\widetilde{\lambda}_Q(y)\,\mathrm{d}y = \int_0^S \mathbb{I}_{[0,\,t]}(y)\left[\lambda(y)/\widetilde{\lambda}_Q(y)\right]\widetilde{\lambda}_Q(y)\,\mathrm{d}y = \int_0^t \lambda(y)\,\mathrm{d}y = \mu(t). \tag{S.10}$$

Because $Q \geq Q^*$ and $t \in [0,S]$ were selected arbitrarily, Equations (S.9) and (S.10) together with Lemma 2 ensure the conclusion of Theorem 1. □

Proof of Theorem 2. In the following analysis, the main idea is based on showing the desired results for an "ideal" arrival process $\{\ddot{N}(t) : t \in [0,S]\}$ that is obtained from the ERP $\{N^\circ(u) : u \geq 0\}$ (at least in principle) by inversion of the given mean-value function $\mu(t)$. Thus for each $Q \geq Q^*$, the finite-horizon processes $\{\widetilde{N}_Q(t) : t \in [0,S]\}$ and $\{\ddot{N}(t) : t \in [0,S]\}$ are derived from the same infinite-horizon ERP $\{N^\circ(u) : u \geq 0\}$. With this setup, we show that for each $t \in [0,S]$ the CIATA-Ph–generated random variable $N_Q(t)$ converges almost surely to $\ddot{N}(t)$ as $Q \to \infty$. Next we show that the random variables $\{[N_Q(t)]^2 : Q \geq Q^*\}$ are uniformly integrable for each $t \in [0,S]$. Exploiting the last two results and an asymptotic expansion for $\mathrm{Var}[N^\circ(u)]$ as $u \to \infty$, we show that this asymptotic expansion is also applicable to $\mathrm{Var}[\ddot{N}(t)]$ and $\mathrm{Var}[N_Q(t)]$ as $t \to \infty$. The latter result and Theorem 1 ensure that Theorem 2 holds.

In the first part of the proof, we show the following uniform-convergence property.

PROPOSITION 3. *(Asymptotic accuracy of the majorizing rate and mean-value functions) If Assumptions 1 and 2 hold, then*

$$\widetilde{\lambda}_Q(t) \downarrow \lambda(t) \ \ \textit{uniformly on} \ \ [0,S] \ \ \textit{as} \ \ Q \to \infty, \tag{S.11}$$

and

$$\widetilde{\mu}_Q(t) \downarrow \mu(t) \ \ \textit{uniformly on} \ \ [0,S] \ \ \textit{as} \ \ Q \to \infty. \tag{S.12}$$

Proof of Proposition 3. With $\mathscr{L}$ and $\mathscr{M}$ respectively defined by Equations (3) and (5), we prove the desired uniform-convergence properties on $[\xi_j^\dagger, \xi_j^\ddagger] \subset \mathscr{M}$ for $j \in \{1,\ldots,M\}$; these properties clearly hold on $(\zeta_\ell^\dagger, \zeta_\ell^\ddagger) \subset \mathscr{L}$ for $\ell \in \{1,\ldots,L\}$. To simplify the discussion, we pick $j \in \{1,\ldots,M\}$ arbitrarily. For $Q \geq Q^*$, let $\delta_{Q,j} = \sup\{\widetilde{\lambda}_Q(t) - \lambda(t) : t \in [\xi_j^\dagger, \xi_j^\ddagger]\}$; and let $t^\#_{Q,j}$ denote a point in $[\xi_j^\dagger, \xi_j^\ddagger]$ such that $\widetilde{\lambda}_Q(t^\#_{Q,j}) - \lambda(t^\#_{Q,j}) = \delta_{Q,j}$. The existence of the points $\{t^\#_{Q,j} : Q \geq Q^*\}$ follows from the constructed properties (ii) and (v) of $\lambda(t)$ on $[\xi_j^\dagger, \xi_j^\ddagger]$ as detailed in §3.2, the continuity of $\lambda(t)$ on $[\xi_j^\dagger, \xi_j^\ddagger]$, and the extreme value theorem. By Assumption 2 and the Heine theorem (Apostol 1974), $\lambda(t)$ is uniformly continuous on $[\xi_j^\dagger, \xi_j^\ddagger]$. For each $Q \geq Q^*$, the constructed property (iii) of the corresponding partition of $[\xi_j^\dagger, \xi_j^\ddagger]$ ensures that the associated subinterval of $[\xi_j^\dagger, \xi_j^\ddagger]$ containing $t^\#_{Q,j}$ has length not exceeding $(\xi_j^\ddagger - \xi_j^\dagger)/Q$; moreover by the constructed property (v) of $\lambda(t)$ and $\widetilde{\lambda}_Q(t)$, on that subinterval $\widetilde{\lambda}_Q(t)$ equals the maximum value of $\lambda(t)$ restricted to the subinterval. From the latter two results and the constructed property (iv) of successive partitions of $[\xi_j^\dagger, \xi_j^\ddagger]$ for $Q = Q^*, Q^*+1, \ldots,$ it follows that $\lim_{Q\to\infty} \delta_Q = \lim_{Q\to\infty} \widetilde{\lambda}_Q(t^\#_Q) - \lambda(t^\#_Q) = 0$, which is equivalent to

Equation (S.11). Equation (S.12) follows immediately from (S.11), (7), and Theorem 9.8 of Apostol (1974). □

In the second part of the proof of Theorem 2 we show that at each $t \in [0,S]$, we have $\widetilde{N}_Q(t) \xrightarrow[Q\to\infty]{\text{a.s.}} \ddot{N}(t)$, where $\xrightarrow[Q\to\infty]{\text{a.s.}}$ denotes almost-sure convergence as $Q \to \infty$. (In some cases, for clarity we use the shorthand phrase "w.p. 1" to mean "with probability 1" or "almost surely.") The ERP $\{N^\circ(u) : u \geq 0\}$ has stationary increments by Theorem 3.5.2 of Ross (1996). Since the c.d.f.'s $G_e(x)$ and $G(x)$ are continuous on $\mathbb{R}_+$ with $G(0) = G_e(0) = 0$, Equation (3.8.4) of Ross (1996) implies that this ERP is a regular point process. Therefore for each $u \in [0,\infty)$, the event $\{2 \text{ or more renewals simultaneously occur at any } w \in [0,u]\}$ has probability 0; see p. 151 of Ross (1996). The latter property ensures that for each $u > 0$, with probability 1 the random function $\{N^\circ(w) : w \in [0,u]\}$ is cadlag—i.e., on $[0,u]$ this function is right-continuous and has left-hand limits (Whitt 2002)—such that all its jumps are of size 1. Thus Theorem 2.1 of Billingsley (1995) implies that the event

$$\mathscr{H} \equiv \{N^\circ(u) \text{ is cadlag on } [0,\infty) \text{ with all jumps of size } 1\} \text{ has } \Pr\{\mathscr{H}\} = 1\,. \tag{S.13}$$

Lemma 1 and Equation (S.13) imply that

$$\text{For each } Q \geq Q^* \text{ and } t \in [0,S],\ \text{w.p. } 1\ \ \widetilde{N}_Q(t) = N^\circ[\widetilde{\mu}_Q(t)] \ \text{ and } \ \ddot{N}(t) = N^\circ[\mu(t)]\,. \tag{S.14}$$

Equations (S.12), (S.13), and (S.14) imply that

$$\text{At each } t \in [0,S],\ \text{w.p. } 1\ \ \widetilde{N}_Q(t) = N^\circ[\widetilde{\mu}_Q(t)] \underset{Q\to\infty}{\longrightarrow} N^\circ[\mu(t)] = \ddot{N}(t)\,. \tag{S.15}$$

In the third part of the proof, we show that at each $t \in [0,S]$, the difference $N_Q(t) - \widetilde{N}_Q(t) \xrightarrow[Q\to\infty]{\text{a.s.}} 0$. To simplify the argument from this point to Equation (S.27), we select a fixed, arbitrary time $t \in [0,S]$. We define the event $\mathscr{G}_n \equiv \{\ddot{N}(t) = n\} \cap \mathscr{H}$ for $n \geq 0$. Since $G(0) = 0$ and $G(x) < 1$ for $x > 0$, we have $G_e(0) = 0$ and $G_e(x) < 1$ for $x > 0$ so that $\Pr\{\mathscr{G}_n\} > 0$ for $n \geq 0$. Equation (S.15) ensures that

$$\text{Given } \mathscr{G}_n \text{ where } n \geq 0,\ \ \widetilde{N}_Q(t) \xrightarrow[Q\to\infty]{\text{a.s.}} n\,. \tag{S.16}$$

Next we account for the set $\mathscr{Z}$ of zeros of $\lambda(y)$ in $[0,t]$. Let $\mathscr{L}_0$ denote the union of the nonoverlapping open intervals in $[0,t]$ on which $\lambda(y) = 0$. By Assumption 1, $\mathscr{L}_0$ is the union of at most a finite number of such intervals; and similarly $[0,t] \setminus \mathscr{L}_0$ is the union of at most a finite number of nonoverlapping closed intervals, each with a partition such that $\lambda(y)$ is quasiconcave on every subinterval of the partition (including any subintervals on which $\lambda(y)$ is a positive constant). Taken over all the nonoverlapping closed intervals contained in $[0,S] \setminus \mathscr{L}_0$, let $\mathbb{V}$ denote the finite set that is the union of the associated partitions.

We show $\lambda(y) > 0$ on $\big([0,t]\setminus\mathscr{L}_0\big)\setminus\mathbb{V}$. Assume on the contrary that $\lambda(y)$ has a zero $y_0 \in \big([0,t]\setminus\mathscr{L}_0\big)\setminus\mathbb{V}$. For each neighborhood (a,b) of y_0 contained in $\big([0,t]\setminus\mathscr{L}_0\big)\setminus\mathbb{V}$, one of the following four cases must hold: (i) $\lambda(a) > 0$ and $\lambda(b) > 0$; (ii) $\lambda(a) > 0$ and $\lambda(b) = 0$; (iii) $\lambda(a) = 0$ and $\lambda(b) > 0$; or (iv) $\lambda(a) = 0$ and $\lambda(b) = 0$. If case (i) holds for some (a,b), then $\lambda(y)$ is not quasiconcave on $(a,b) \subset \big([0,t]\setminus\mathscr{L}_0\big)\setminus\mathbb{V}$, which is impossible by Assumption 1. If case (ii) holds for some (a,b), then $\lambda(y) = 0$ for $y \in (y_0,b)$, which is impossible because $(y_0,b) \subset \big([0,t]\setminus\mathscr{L}_0\big)\setminus\mathbb{V}$. Similarly if case (iii) holds for some (a,b), then $\lambda(y) = 0$ for $y \in (a,y_0)$, which is impossible because $(a,y_0) \subset \big([0,t]\setminus\mathscr{L}_0\big)\setminus\mathbb{V}$. Finally if case (iv) holds for some (a,b), then it follows that $(a,b) \subset \mathscr{L}_0$, which is impossible because $(a,b) \subset \big([0,t]\setminus\mathscr{L}_0\big)\setminus\mathbb{V}$. Thus all possibilities are exhausted; and this contradiction ensures $\mathscr{Z} \subset \mathscr{L}_0 \cup \mathbb{V}$ so that $\mu(\mathscr{Z})$ consists of at most a finite number of points in $[0,\mu(t)]$.

Since the renewal epochs are $\{S_j^\circ : j \geq 1\}$ are continuous random variables, for each $n \geq 1$ we have $\Pr\big\{S_j^\circ \notin \mu(\mathscr{Z}) \text{ for } 1 \leq j \leq n \,\big|\, \mathscr{G}_n\big\} = 1$. For $n \geq 1$, we let $\mathscr{G}_n' \equiv \mathscr{G}_n \cap \big\{S_j^\circ \notin \mu(\mathscr{Z}) \text{ for } 1 \leq j \leq n\big\}$; and we take $\mathscr{G}_0' \equiv \mathscr{G}_0$ so that $\Pr\{\mathscr{G}_n'|\mathscr{G}_n\} = 1$ and $\Pr\{\mathscr{G}_n'\} = \Pr\{\mathscr{G}_n\} > 0$ for $n \geq 0$. It follows that the arrival epochs $\{\ddot{S}_j = \mu^{-1}(S_j^\circ) : j \geq 1\}$ of the ideal arrival process have the following property:

$$\text{Given } \mathscr{G}_n' \text{ where } n \geq 1,\ \text{w.p. 1}\ \ \lambda(\ddot{S}_j) > 0 \ \text{ for }\ 1 \leq j \leq n. \tag{S.17}$$

For each $Q \geq Q^*$, the constructed properties of $\widetilde{\lambda}_Q(y)$ on $[0,t]$ ensure that $\widetilde{\lambda}_Q(y) > 0$ on $[0,t]$ and $\widetilde{\mu}_Q(y)$ is continuous and strictly increasing on $[0,t]$; thus by Assumptions 1 and 2, we have

$$\widetilde{\mu}_Q^{-1}(w) \uparrow \mu^{-1}(w) \ \text{ for almost all }\ w \in [0,\mu(t)] \ \text{ as }\ Q \to \infty. \tag{S.18}$$

Equation (S.18) implies that given $\mathscr{G}_n'$ where $n \geq 1$, the arrival epochs $\widetilde{S}_{j,Q} \equiv \widetilde{\mu}_Q^{-1}(S_j^\circ)$ for $1 \leq j \leq n$ and $Q \geq Q^*$ satisfy the relation

$$\text{Given } \mathscr{G}_n' \text{ where } n \geq 1,\ \text{w.p. 1}\ \ \widetilde{S}_{j,Q} \uparrow \ddot{S}_j \ \text{ as }\ Q \to \infty \ \text{ for }\ 1 \leq j \leq n. \tag{S.19}$$

Since $\lambda(y)$ is continuous at each $y \in [0,t]$ by Assumption 2, it follows from Equations (S.11), (S.17), and (S.19) together with Theorem 4.4 of Rudin (1964) that

$$\text{Given } \mathscr{G}_n' \text{ where } n \geq 1,\ \text{w.p. 1}\ \ \lim_{Q\to\infty} \frac{\lambda(\widetilde{S}_{j,Q})}{\widetilde{\lambda}_Q(\widetilde{S}_{j,Q})} = \frac{\lambda(\ddot{S}_j)}{\lambda(\ddot{S}_j)} = 1 \ \text{ for }\ 1 \leq j \leq n. \tag{S.20}$$

Continuing the third part of the proof, we let $\ddot{\mathscr{P}}(z) \equiv \sum_{n=0}^{\infty} \Pr\{\ddot{N}(t) = n\} z^n$ for $-1 \leq z \leq 1$ denote the probability generating function of the random variable $\ddot{N}(t)$. Pick $\varepsilon' \in (0,1)$ arbitrarily small. Since $\ddot{\mathscr{P}}(1) = 1$ and $\ddot{\mathscr{P}}(z)$ is continuous on $[-1,1]$, we can find $\nu \in (0,1)$ sufficiently small so that $\ddot{\mathscr{P}}(z) > 1 - \varepsilon'/2$ for $z \in [1-\nu, 1]$. Theorem 9.20 of Apostol (1974) implies that

$$\sum_{n=0}^{K} \Pr\{\ddot{N}(t) = n\} z^n \uparrow \ddot{\mathscr{P}}(z) \ \text{ uniformly on }\ [1-\nu, 1-\nu/2] \ \text{ as }\ K \to \infty; \tag{S.21}$$

and it follows from Equation (S.21) that there is a positive integer K^* such that

$$\sum_{n=0}^{K} \Pr\{\ddot{N}(t)=n\} z^n > \ddot{\mathscr{P}}(z) - \varepsilon'/2 \ \text{ for all } \ z \in [1-\nu, 1-\nu/2] \ \text{ when } \ K \geq K^* . \tag{S.22}$$

Equations (S.11), (S.18), and (S.20) imply that

$$\left.\begin{array}{l} \text{Given } \mathscr{G}_n' \text{ where } n \geq 1\,, \text{ w.p. 1 } \ \lim_{Q\to\infty} \lambda(\widetilde{S}_{j,Q}) \big/ \widetilde{\lambda}_Q(\widetilde{S}_{j,Q}) \geq 1-\nu \ \text{ for } \ 1 \leq j \leq n \\ \text{Given } \mathscr{G}_0', \text{ w.p. 1 } \ \lim_{Q\to\infty} N_Q(t) = \lim_{Q\to\infty} \widetilde{N}_Q(t) = 0 \end{array}\right\} . \tag{S.23}$$

Next we choose $\delta' \in (0,1)$ arbitrarily small, where without loss of generality we may assume that $\delta' < 1$. Given $\mathscr{G}_n'$ with $n \geq 1$, each of the arrival epochs $\{\widetilde{S}_{j,Q} : 1 \leq j \leq n\}$ is accepted or rejected independently for every $Q \geq Q^*$; and since $\delta' < 1$, Equation (S.23) implies that

$$\Pr\left\{ \limsup_{Q\to\infty} \left| N_Q(t) - \widetilde{N}_Q(t) \right| < \delta' \,\middle|\, \mathscr{G}_n' \right\} \geq (1-\nu)^n \ \text{ for } \ n \geq 0 . \tag{S.24}$$

Because $\Pr\{\mathscr{H}\} = 1$ and $\Pr\{\mathscr{G}_n' | \mathscr{G}_n\} = 1$ for $n \geq 0$, we have

$$\Pr\{\ddot{N}(t)=n\} = \Pr\{\mathscr{G}_n\} = \Pr\{\mathscr{G}_n'\} \ \text{ and } \ \Pr\{\mathscr{G}_n \setminus \mathscr{G}_n'\} = 0 \ \text{ for } \ n \geq 0 ; \tag{S.25}$$

and from Equation (S.25) it follows that

$$\begin{aligned} \Pr\left\{ \limsup_{Q\to\infty} \left| N_Q(t) - \widetilde{N}_Q(t) \right| < \delta' \,\middle|\, \ddot{N}(t)=n \right\} &= \Pr\left\{ \limsup_{Q\to\infty} \left| N_Q(t) - \widetilde{N}_Q(t) \right| < \delta' \,\middle|\, \mathscr{G}_n \right\} \\ &= \Pr\left\{ \limsup_{Q\to\infty} \left| N_Q(t) - \widetilde{N}_Q(t) \right| < \delta' \,\middle|\, \mathscr{G}_n' \right\} \ \text{ for } \ n \geq 0 . \end{aligned} \tag{S.26}$$

Equations (S.22), (S.24), and (S.26) imply that

$$\begin{aligned} \Pr\left\{ \limsup_{Q\to\infty} \left| N_Q(t) - \widetilde{N}_Q(t) \right| < \delta' \right\} &= \sum_{n=0}^{\infty} \Pr\{\ddot{N}(t)=n\} \Pr\left\{ \limsup_{Q\to\infty} \left| N_Q(t) - \widetilde{N}_Q(t) \right| < \delta' \,\middle|\, \ddot{N}(t)=n \right\} \\ &\geq \sum_{n=0}^{K^*} \Pr\{\ddot{N}(t)=n\} \Pr\left\{ \limsup_{Q\to\infty} \left| N_Q(t) - \widetilde{N}_Q(t) \right| < \delta' \,\middle|\, \ddot{N}(t)=n \right\} \\ &= \sum_{n=0}^{K^*} \Pr\{\ddot{N}(t)=n\} \Pr\left\{ \limsup_{Q\to\infty} \left| N_Q(t) - \widetilde{N}_Q(t) \right| < \delta' \,\middle|\, \mathscr{G}_n' \right\} \\ &\geq \sum_{n=0}^{K^*} \Pr\{\ddot{N}(t)=n\} (1-\nu)^n \geq \ddot{\mathscr{P}}(1-\nu) - \varepsilon'/2 \\ &> [1-\varepsilon'/2] - \varepsilon'/2 = 1 - \varepsilon' , \end{aligned} \tag{S.27}$$

Since $t \in [0,S]$ is arbitrary and $\delta', \varepsilon' \in (0,1)$ are arbitrary, Equation (S.27) implies that

$$\text{At each } t \in [0,S], \ \ N_Q(t) - \widetilde{N}_Q(t) \xrightarrow[Q\to\infty]{\text{a.s.}} 0 . \tag{S.28}$$

It follows from Equations (S.15) and (S.28) that

$$\text{At each } t \in [0,S], \ \ N_Q(t) = \widetilde{N}_Q(t) + [N_Q(t) - \widetilde{N}_Q(t)] \xrightarrow[Q\to\infty]{\text{a.s.}} \ddot{N}(t) . \tag{S.29}$$

In the fourth part of the proof, we start by showing that the random variables $\{[N_Q(t)]^2 : Q \geq Q^*\}$ are uniformly integrable for each $t \in [0,S]$. By Assumption 1, Lemma 1, and Equation (S.14), we see that

$$\text{At each } t \in [0,S],\ \text{w.p. 1}\ \ [N_Q(t)]^2 \leq [\widetilde{N}_Q(t)]^2 = \big\{N^\circ[\widetilde{\mu}_Q(t)]\big\}^2 \leq [N^\circ(\lambda^* t)]^2 \ \text{ for }\ Q \geq Q^*. \tag{S.30}$$

Since $\theta_3 \equiv \mathrm{E}\big[(X_2^\circ)^3\big] < \infty$ for the interrenewal distributions used in CIATA-Ph, Equation (18) on p. 58 of Cox (1962) implies that

$$\mathrm{E}\big\{[N^\circ(\lambda^* t)]^2\big\} < \infty \ \text{ for each }\ t \in [0,S]. \tag{S.31}$$

Equations (S.30)–(S.31) and the dominated convergence theorem ensure that the random variables $\{[N_Q(t)]^2 : Q \geq Q^*\}$ are uniformly integrable for each $t \in [0,S]$.

Finally we observe from Equation (S.29) that

$$\text{At each } t \in [0,S],\ \ [N_Q(t)]^2 \xrightarrow[Q\to\infty]{\text{a.s.}} [\ddot{N}(t)]^2; \tag{S.32}$$

and thus Theorem 25.2 of Billingsley (1995) ensures that

$$\text{At each } t \in [0,S],\ \ [N_Q(t)]^2 \xrightarrow[Q\to\infty]{d} [\ddot{N}(t)]^2 \tag{S.33}$$

where $\xrightarrow[Q\to\infty]{d}$ denotes convergence in distribution as $Q \to \infty$. Uniform integrability of the $\{[N_Q(t)]^2 : Q \geq Q^*\}$, Equation (S.33), and Theorem 25.12 of Billingsley (1995) imply that

$$\lim_{Q\to\infty} \mathrm{E}\big\{[N_Q(t)]^2\big\} = \mathrm{E}\big\{[\ddot{N}(t)]^2\big\} < \infty \ \text{ for each }\ t \in [0,S]; \tag{S.34}$$

and by Theorem 1 and Equation (S.34), we have

$$\lim_{Q\to\infty} \mathrm{Var}[N_Q(t)] = \mathrm{Var}[\ddot{N}(t)] < \infty \ \text{ for each }\ t \in [0,S]. \tag{S.35}$$

Since $\theta_3 < \infty$, Equation (18) on p. 58 of Cox (1962) implies that the infinite-horizon ERP $\{N^\circ(u) : u \geq 0\}$ has the following variance expansion,

$$\mathrm{Var}[N^\circ(u)] = Cu + \theta^* + o(1) \ \text{ as }\ u \to \infty, \tag{S.36}$$

where θ^* is defined by Equation (9), and in general the term $o[h(u)]$ denotes a function $g(u)$ such that $g(u)/h(u) \to 0$ as $u \to \infty$. From Equations (10), (S.14), and (S.36) it follows that

$$\mathrm{Var}[\ddot{N}(t)] = \mathrm{Var}\{N^\circ[\mu(t)]\} = C\mu(t) + \theta^* + o(1) \ \text{ as }\ S \to \infty \ \text{ and }\ t \to \infty. \tag{S.37}$$

The final conclusions (11) and (12) follow from Theorem 1 and Equations (10), (S.35), and (S.37). □

S2. Proofs of Supporting Results

S2.1 Preliminaries on Equilibrium Renewal Processes

Consider a set of nonnegative interarrival times $\{X_n^\circ :\ n = 1,2,\ldots\}$, where the subset $\{X_n^\circ :\ n = 2,3,\ldots\}$ are independent and identically distributed with cumulative distribution function G, while X_1°, the time until the first event, may have a different distribution. As usual, we let S_n° denote the time of the nth event, and we let $N^\circ(u)$ denote the number of events observed on or before time u. Such a renewal process $\{N^\circ(t) : t \geq 0\}$ is called a delayed renewal process; and it is generated by taking $\{X_j^\circ : n = 2,3,\ldots\} \overset{\text{i.i.d.}}{\sim} G$. We assume that the noncentral moments $\theta_\ell \equiv \mathrm{E}\big[\left(X_2^\circ\right)^\ell\big]$ for $\ell = 1,2,3$, are all finite so that for $j \geq 2$, the interrenewal times have $\mathrm{E}[X_j^\circ] = \theta_1$, variance $\mathrm{Var}[X_j^\circ] = \theta_2 - \theta_1^2$, and coefficient of variation $\mathrm{CV}[X_j^\circ] = (\theta_2 - \theta_1^2)^{1/2}/\theta_1$.

If X_1° has the equilibrium distribution associated with G, specifically,

$$G_e(t) \equiv \Pr\{X_1^\circ \leq t\} = \frac{1}{\theta_1}\int_0^t [1 - G(u)]\,\mathrm{d}u \text{ for } t \geq 0,$$

then $\{N^\circ(t) : t \geq 0\}$ is an ERP, so that by Equation (3) on p. 46 of Cox (1962), we have

$$\mathrm{E}[N^\circ(t)] = \frac{t}{\theta_1} \text{ for } t \geq 0. \tag{S.38}$$

Moreover by Equation (18) on p. 58 of Cox (1962), we have

$$\mathrm{Var}[N^\circ(t)] = \left(\frac{\theta_2 - \theta_1^2}{\theta_1^3}\right)t + \frac{1}{6} + \frac{(\theta_2 - \theta_1^2)^2}{2\theta_1^4} - \frac{\theta_3}{3\theta_1^3} + o(1) \text{ as } t \to \infty\,.$$

S2.2 Proof of Proposition 1

For $C > 1$, we consider the two-phase balanced-means hyperexponential distribution with c.d.f.

$$\begin{aligned} G(x) &= p(1 - e^{-2px}) + (1-p)(1 - e^{-2(1-p)x}) \\ &= 1 - pe^{-2px} - (1-p)e^{-2(1-p)x} \quad \text{for } x \geq 0\,, \end{aligned} \tag{S.39}$$

so that the interrenewal times $\{X_i^\circ : i = 2,3,\ldots\} \overset{\text{i.i.d.}}{\sim} G(x)$ have the moments $\mathrm{E}[X_i^\circ] = 1$ and $\mathrm{Var}[X_i^\circ] = C$. For $i = 2,3,\ldots$, we have

$$\mathrm{E}[(X_i^\circ)^2] = p\mathrm{E}[Y_1^2] + (1-p)\mathrm{E}[Y_2^2] = \frac{1}{2p} + \frac{1}{2(1-p)}\,,$$

where Y_1 is an exponentially distributed random variable with $\mathrm{E}[Y_1] = 1/(2p)$ and Y_2 is an exponentially distributed random variable with $\mathrm{E}[Y_2] = 1/[2(1-p)]$ so that we have

$$\mathrm{Var}(X_i^\circ) = \mathrm{E}[(X_i^\circ)^2] - \mathrm{E}^2[X_i^\circ] = \frac{1}{2p} + \frac{1}{2(1-p)} - 1.$$

To achieve $\mathrm{CV}^2[X_i^\circ] = C$ for $i = 2,3,\ldots$ for a given value of $C \in [1,\infty)$, we must solve the following quadratic equation for $p \in (0,1)$:

$$\frac{1 - 2p(1-p)}{2p(1-p)} = C.$$

It follows that we have

$$p = \frac{1+C \pm \sqrt{C^2-1}}{2(1+C)}. \tag{S.40}$$

Note that both roots of Equation (S.40) belong to the unit interval $(0,1)$ and effectively yield the same 2-phase balanced-means hyperexponential distribution $G(x) = F_{H_2}[x;\, p, 2p, 2(1-p)]$. Finally the c.d.f. of the first interarrival time is

$$G_e(t) = \frac{1}{\mathrm{E}[X_2^\circ]}\int_0^t [1-G(u)]\,\mathrm{d}u = 1 - \tfrac{1}{2}e^{-2pt} - \tfrac{1}{2}e^{-2(1-p)t} = F_{H_2}[x;\, 1/2, 2p, 2(1-p)] \text{ for } t \geq 0. \quad \square$$

S2.3 Proof of Proposition 2

When $C < 1$, we want the interrenewal times $\{X_i^\circ, i = 2,3,\ldots\}$ to follow a hyper-Erlang distribution with $\mathrm{E}[X_i^\circ] = 1$ and $\mathrm{Var}[X_i^\circ] = C < 1$. We take $k = \lceil 1/C \rceil$ so that $k \geq 2$. For $i \geq 2$, we have

$$\mathrm{E}[X_i^\circ] = p[(k-1)\beta] + (1-p)(k\beta) = (k-p)\beta = 1,$$

so that we must have $\beta = 1/(k-p)$. Moreover for $i \geq 2$, we have

$$\mathrm{E}[(X_i^\circ)^2] = p\mathrm{E}[Y_{k-1}^2] + (1-p)\mathrm{E}[Y_k^2] = p[(k-1)k\beta^2] + (1-p)[k(k+1)\beta^2] = k(k+1-2p)\beta^2,$$

where $Y_{k-1} \sim F_{\mathrm{Er}}(x;\, k-1, \beta)$ and $Y_k \sim F_{\mathrm{Er}}(x;\, k, \beta)$ so that

$$\mathrm{Var}[X_i^\circ] = \mathrm{E}[(X_i^\circ)^2] - (\mathrm{E}[X_i^\circ])^2 = k(k+1-2p)\beta^2 - [\beta(k-p)]^2 = \frac{k-p^2}{\beta^2}.$$

To achieve $\mathrm{CV}^2[X_i^\circ] = C$ for $i \geq 2$, where $C \in (0,1)$ is given, we must solve the following quadratic equation for p:

$$C = \frac{k-p^2}{(k-p)^2}.$$

Solving for p in the above equation, we have

$$p = \frac{Ck \pm \sqrt{k(1+C) - k^2 C}}{1+C}. \tag{S.41}$$

Note that in Equation (S.41) the solution achieved by taking the root corresponding to the plus sign yields a value of p outside the unit interval [0,1):

$$\frac{kC + \sqrt{k(1+C) - k^2 C}}{1+C} = \frac{kC + \sqrt{k(1+C-kC)}}{1+C} > \frac{kC + (1+C-kC)}{1+C} = 1$$

since $k \geq 2 > 1 + (1-k)C$. The other solution,

$$p = \frac{kC - \sqrt{k(1+C) - k^2 C}}{1+C},$$

$p \in [0,1)$ for the following reason. Because $k = \lceil 1/C \rceil$, we have $\sqrt{k+kC-k^2C} \le \sqrt{k+kC-k} = \sqrt{Ck} \le Ck$, so that $p \ge 0$. Moreover, $Ck-(1+C) = C(k-1)-1 < 0$ so that $Ck^2 < k(1+C)$; hence it follows that $\sqrt{k+kC-k^2C} > 0$ and $p < 1$.

For for $i \ge 2$, the c.d.f. for the hyper-Erlang interarrival times $\{X_i^\circ\}$ is

$$G(t) = pF_{\mathrm{Er}}[t;k-1,\beta] + (1-p)F_{\mathrm{Er}}[t;k,\beta] \text{ for all } t \tag{S.42}$$

where

$$F_{\mathrm{Er}}(t;k,\beta) = \int_0^t \frac{u^{k-1}e^{-u/\beta}}{(k-1)!\beta^k}\,\mathrm{d}u = 1 - e^{-t/\beta}\sum_{n=0}^{k-1}\frac{(t/\beta)^n}{n!} \quad \text{for } t \ge 0.$$

Finally it follows that for the c.d.f. for the first interarrival time X_1 is given by

$$G_e(t) = \frac{1}{\mathrm{E}[X_2^\circ]}\int_0^t [1-G(x)]\,\mathrm{d}x \tag{S.43}$$

$$= \int_0^t \big[1 - pF_{\mathrm{Er}}(x;k-1,\beta) - (1-p)F_{\mathrm{Er}}(x;k,\beta)\big]\,\mathrm{d}x$$

$$= \big[1 - pF_{\mathrm{Er}}(x;k-1,\beta) - (1-p)F_{\mathrm{Er}}(x;k,\beta)\big]x\Big|_0^t + \int_0^t x\big[pf_{\mathrm{Er}}(x;k-1,\beta) + (1-p)f_{\mathrm{Er}}(x;k,\beta)\big]\,\mathrm{d}x$$

$$= \big[1 - pF_{\mathrm{Er}}(t;k-1,\beta) - (1-p)F_{\mathrm{Er}}(t;k,\beta)\big]t + p\int_0^t x\frac{x^{k-2}e^{-x/\beta}}{(k-2)!\beta^{k-1}}\,\mathrm{d}x + (1-p)\int_0^t x\frac{x^{k-1}e^{-x/\beta}}{(k-1)!\beta^k}\,\mathrm{d}x$$

$$= \big[1 - pF_{\mathrm{Er}}(t;k-1,\beta) - (1-p)F_{\mathrm{Er}}(t;k,\beta)\big]t + p(k-1)\beta F_{\mathrm{Er}}(t;k,\beta) + (1-p)k\beta F_{\mathrm{Er}}(t;k+1,\beta) \tag{S.44}$$

$$= \big[1 - F_{\mathrm{Er}}(t;k-1,\beta)\big]t + F_{\mathrm{Er}}(t;k,\beta) \text{ for all } t \ge 0, \tag{S.45}$$

where the final result (S.45) follows by making the substitutions

$$F_{\mathrm{Er}}(t;k,\beta) = F_{\mathrm{Er}}(t;k-1,\beta) - \frac{e^{-t/\beta}t^{k-1}}{(k-1)!\beta^{k-1}} \text{ and } F_{\mathrm{Er}}(t;k+1,\beta) = F_{\mathrm{Er}}(t;k,\beta) - \frac{e^{-t/\beta}t^k}{k!\beta^k}$$

in Equation (S.44). □

S3. Additional Simulation Experiments

We provide addition simulation experiments to supplement §4 of the main paper. In §S3.1 we provide an additional figure supporting the discussion in §4.4. In §S3.2 we provide additional results for test cases 1 and 3, and we provide full results for test cases 2 and 5. In §S3.3 we compare the results of warm-up times and closeness to furthermore evaluate the effectiveness of CIATA-Ph.

S3.1 Supplementary Results for Case 1 with $Q = 40$

As a supplement to the discussion in §4.4 about effectively assigning a value to Q, Figure S1 depicts the performance of CIATA-Ph in case 1 for $C = 1.5$ and $C = 0.2$ when we take $Q = 40$.

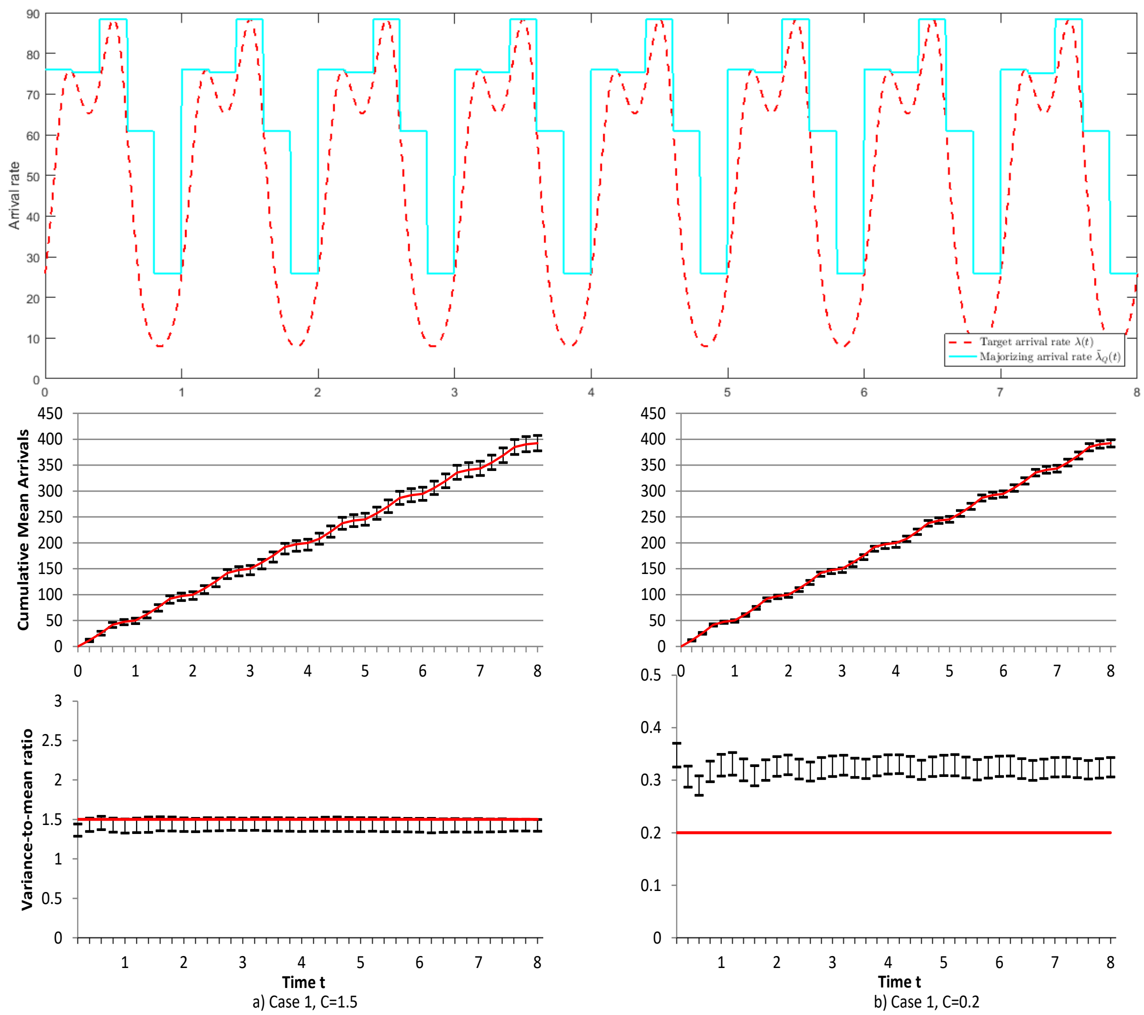

Figure S1 **CIATA-Ph Performance for Case 1 with Dispersion Ratio $C = 1.5, 0.2$ and with $Q = 40$: (i) (Top Panel) the Majorizing Rate Function $\widetilde{\lambda}_Q(t)$; (ii) (Middle Panel) 95% CIs for the Mean-Value Function $\mu(t)$ with $C = 1.5$ (Left) and $C = 0.2$ (Right); and (iii) (Bottom Panel) 95% CIs for $C_Q(t)$ with $C = 1.5$ (Left) and $C = 0.2$ (Right)**

S3.2 Additional Experiments with CIATA-Ph

In this section we expand the performance evaluation results of CIATA-Ph with Case-1 and Case-3 arrival rates (considered in §4) by conducting experiments with $C = 10$ and 0.8. For the corresponding results, see Figure S2 and Table S1 for test process 1; and see Figure S5 and Table S3 for test process 3. We also conduct experiments with CIATA-Ph using the Case-2 and Case-5 arrival rates. In Figures S3 and S4, we report the following for test case 2 with $C = 10, 1.5, 0.8$ and 0.2: (i) the majorizing arrival rate; (ii) 95% CI estimators of $\mu(t)$; and (iii) 95% CI estimators of $C_Q(t)$. The corresponding closeness results are summarized in Table S2. In Figure S6, we report the items (i), (ii) and (iii) for test case 5 with $C = 1.5$ and 0.2; and in Table S4 we report the associated closeness measures for test case 5. We conclude that CIATA-Ph works effectively for all these cases.

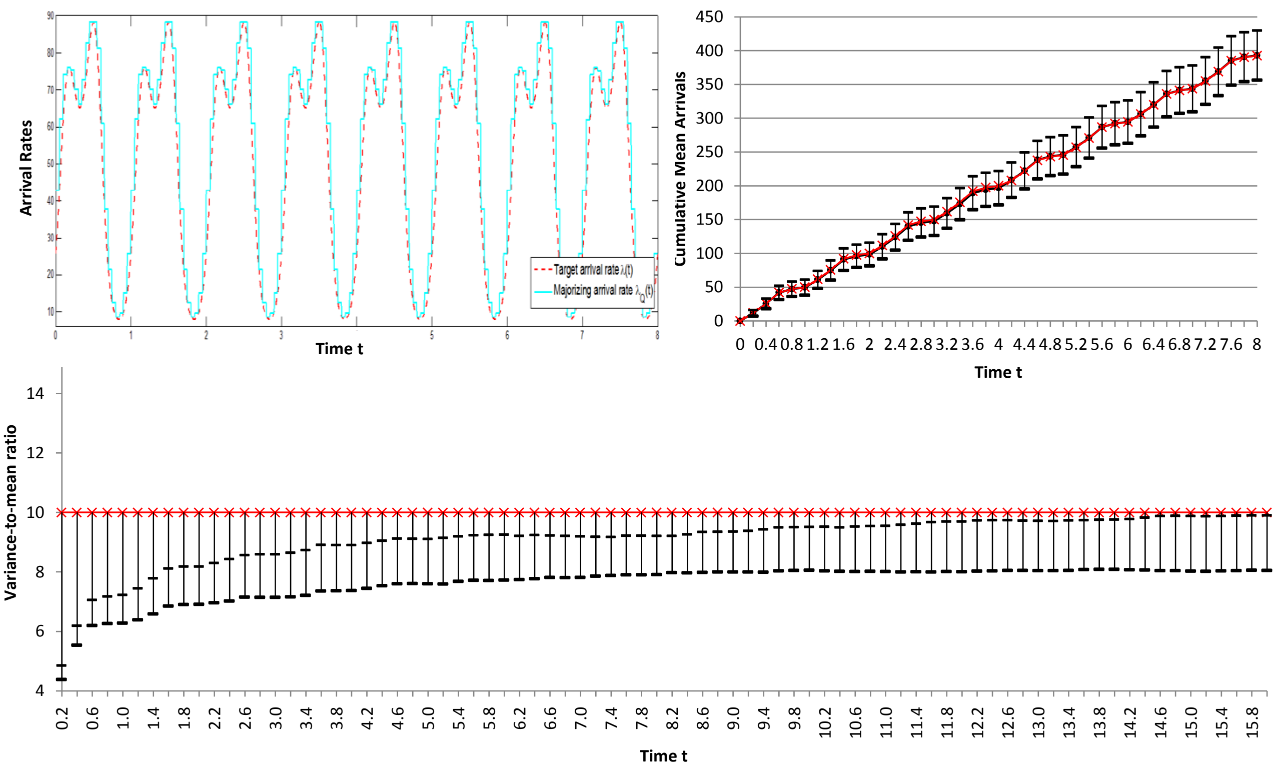

Figure S2 **CIATA-Ph Performance for Case-1 Arrival Rates and Dispersion Ratio** $C = 10$ **and 0.8: (i) (Top)** 95% **CIs for the Mean-Value Function** $\mu(t)$ **with** $C = 10$ **(Left) and** $C = 0.8$ **(Right); and (ii) (Bottom)** 95% **CIs for** $C_Q(t)$ **with** $C = 10$ **(Left) and** $C = 0.8$ **(Right) .**

Table S1 **CIATA-Ph–based Closeness Measures for Case 1 with** $C = 0.8$ **and 10.**

	Case 1			
C	$\Delta_{\mathbb{T}}(\widehat{\mu}_Q)$	$\Delta^*_{\mathbb{T}}(\widehat{\mu}_Q)$	$\Delta_{\mathbb{T}}(\widehat{C}_Q)$	$\Delta^*_{\mathbb{T}}(\widehat{C}_Q)$
0.8	$0.915\% \pm 0.200\%$	2.028%	$1.669\% \pm 0.323\%$	5.163%
10.	$0.957\% \pm 0.190\%$	2.287%	$10.324\% \pm 1.146\%$	38.490%

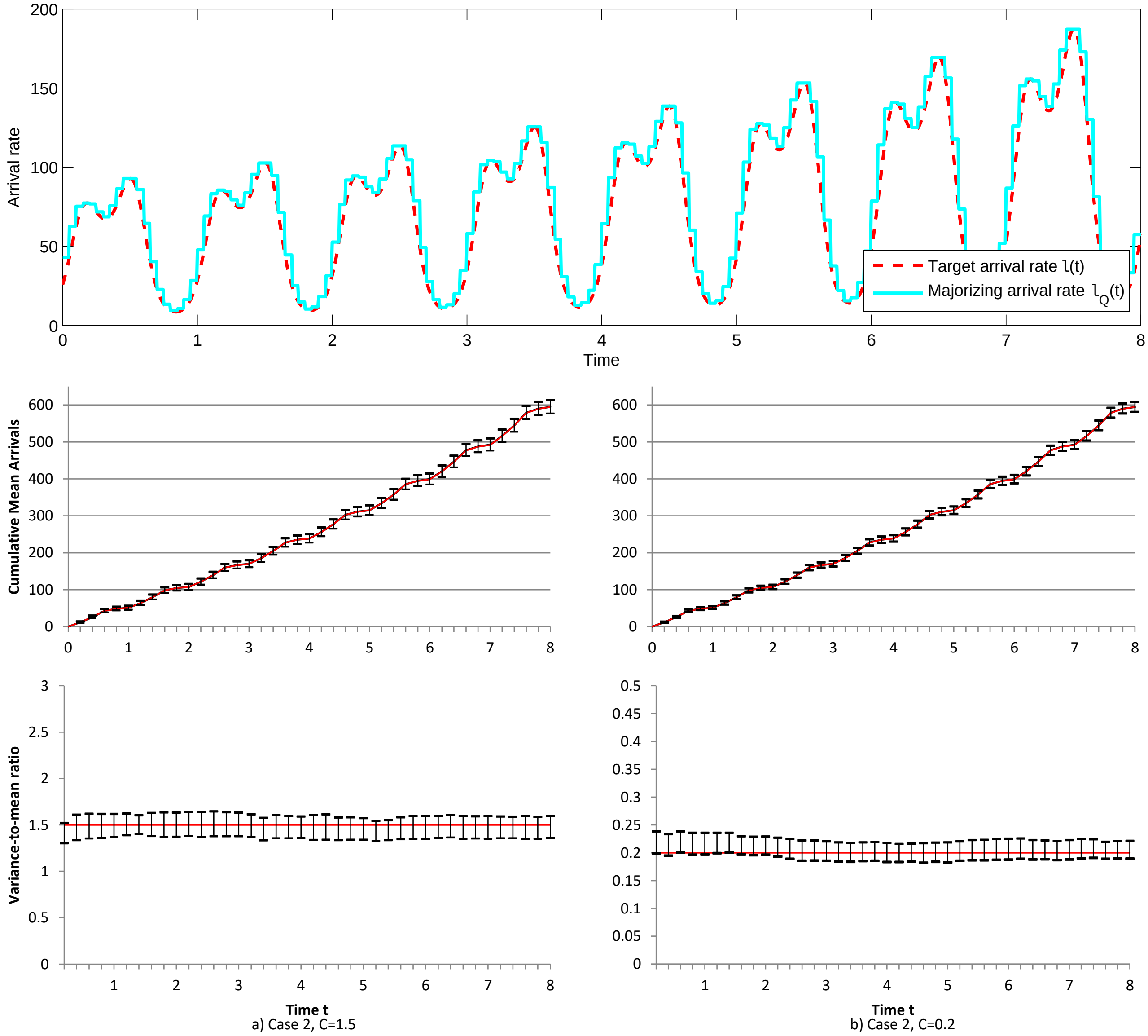

Figure S3 **CIATA-Ph Performance for Case-2 Arrival Rates and Dispersion Ratio $C = 1.5$ and 0.2: (i) (Top) the Majorizing Rate Function $\widetilde{\lambda}_Q(t)$; (ii) (Middle) 95% CIs for the Mean-Value Function $\mu(t)$ with $C = 1.5$ (Left) and $C = 0.2$ (Right); and (iii) (Bottom) 95% CIs for $C_Q(t)$ with $C = 1.5$ (Left) and $C = 0.2$ (Right) .**

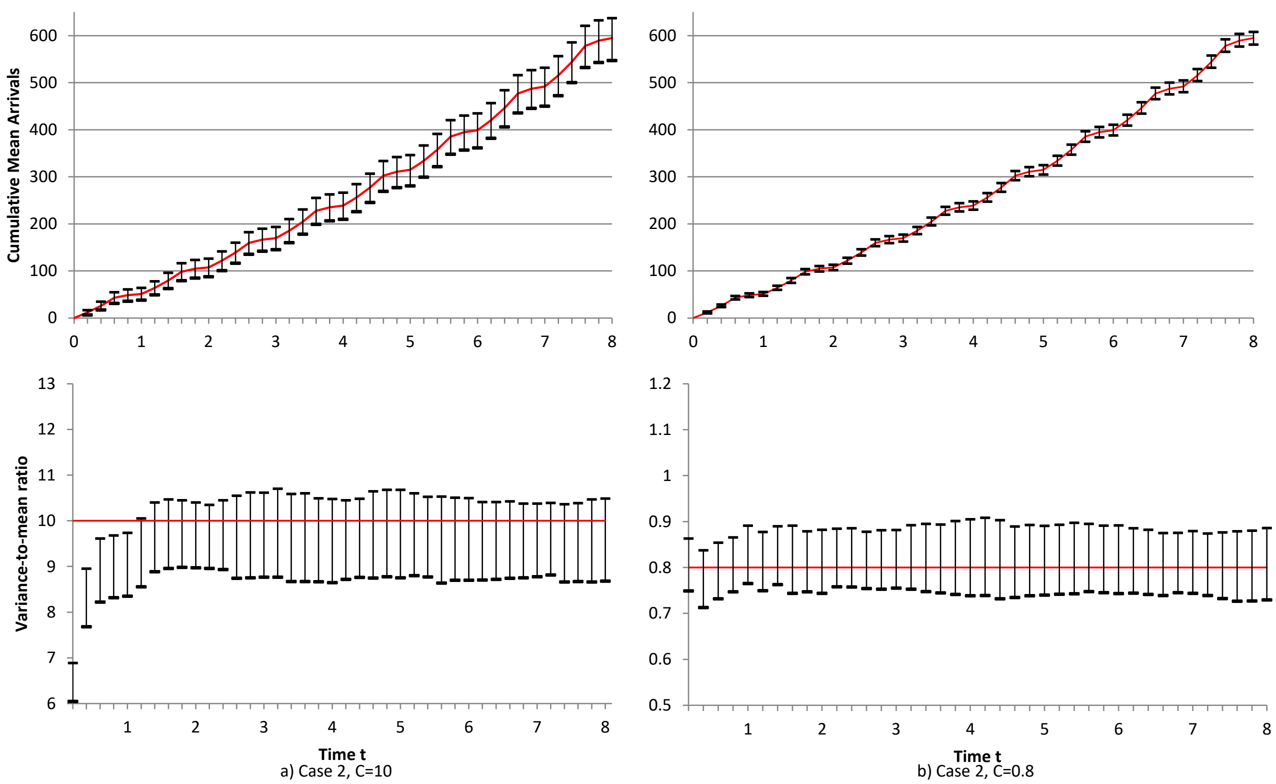

Figure S4 **CIATA-Ph Performance for Case-2 Arrival Rates and Dispersion Ratio** $C = 10$ **and 0.8: (i) (Top)** 95% **CIs for the Mean-Value Function** $\mu(t)$ **with** $C = 10.$ **(Left) and** $C = 0.8$ **(Right); and (ii) (Bottom)** 95% **CIs for** $C_Q(t)$ **with** $C = 1.5$ **(Left) and** $C = 0.2$ **(Right) .**

Table S2 **CIATA-Ph–based Closeness Measures for Case 2.**

Case 2				
C	$\Delta_{\mathbb{T}}(\widehat{\mu}_Q)$	$\Delta^*_{\mathbb{T}}(\widehat{\mu}_Q)$	$\Delta_{\mathbb{T}}(\widehat{C}_Q)$	$\Delta^*_{\mathbb{T}}(\widehat{C}_Q)$
0.2	$0.036\% \pm 0.009\%$	0.166%	$3.449\% \pm 0.634\%$	9.600%
0.8	$0.107\% \pm 0.019\%$	0.418%	$1.609\% \pm 0.266\%$	5.907%
1.5	$0.084\% \pm 0.026\%$	0.767%	$1.609\% \pm 0.266\%$	5.907%
10.	$0.390\% \pm 0.028\%$	0.709%	$5.373\% \pm 1.219\%$	35.302%

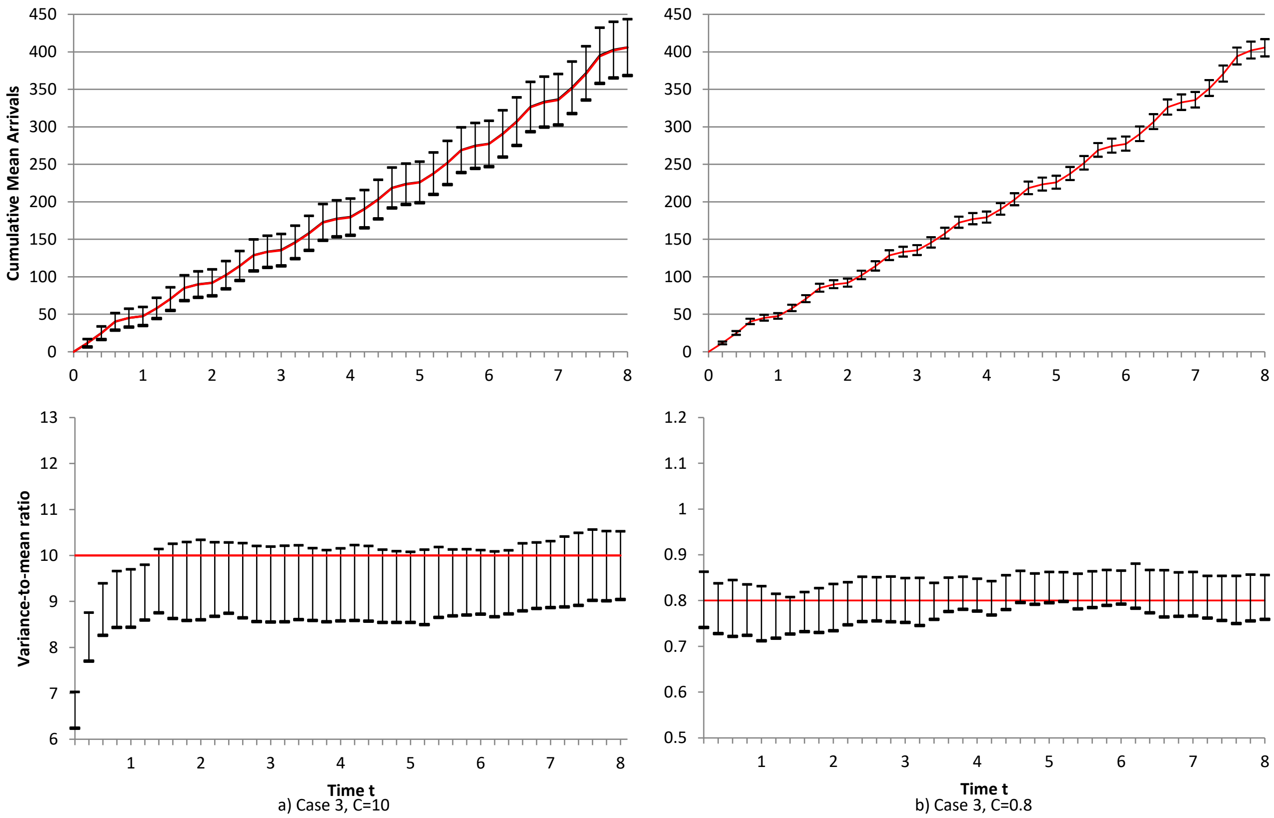

Figure S5 **CIATA-Ph Performance for Case-3 Arrival Rates and Dispersion Ratio $C = 10$ and 0.8: (i) (Top) 95% CIs for the Mean-Value Function $\mu(t)$ with $C = 10$ (Left) and $C = 0.8$ (Right); and (ii) (Bottom) 95% CIs for $C_Q(t)$ with $C = 10$ (Left) and $C = 0.8$ (Right) .**

Table S3 **CIATA-Ph–based Closeness Measures for Case 3 with $C = 0.8$ and 10.**

Case 1				
C	$\Delta_{\mathbb{T}}(\widehat{\mu}_Q)$	$\Delta^*_{\mathbb{T}}(\widehat{\mu}_Q)$	$\Delta_{\mathbb{T}}(\widehat{C}_Q)$	$\Delta^*_{\mathbb{T}}(\widehat{C}_Q)$
0.8	$0.149\% \pm 0.015\%$	0.318%	$2.004\% \pm 0.287\%$	4.188%
10.	$0.240\% \pm 0.024\%$	0.548%	$6.778\% \pm 1.115\%$	33.627%

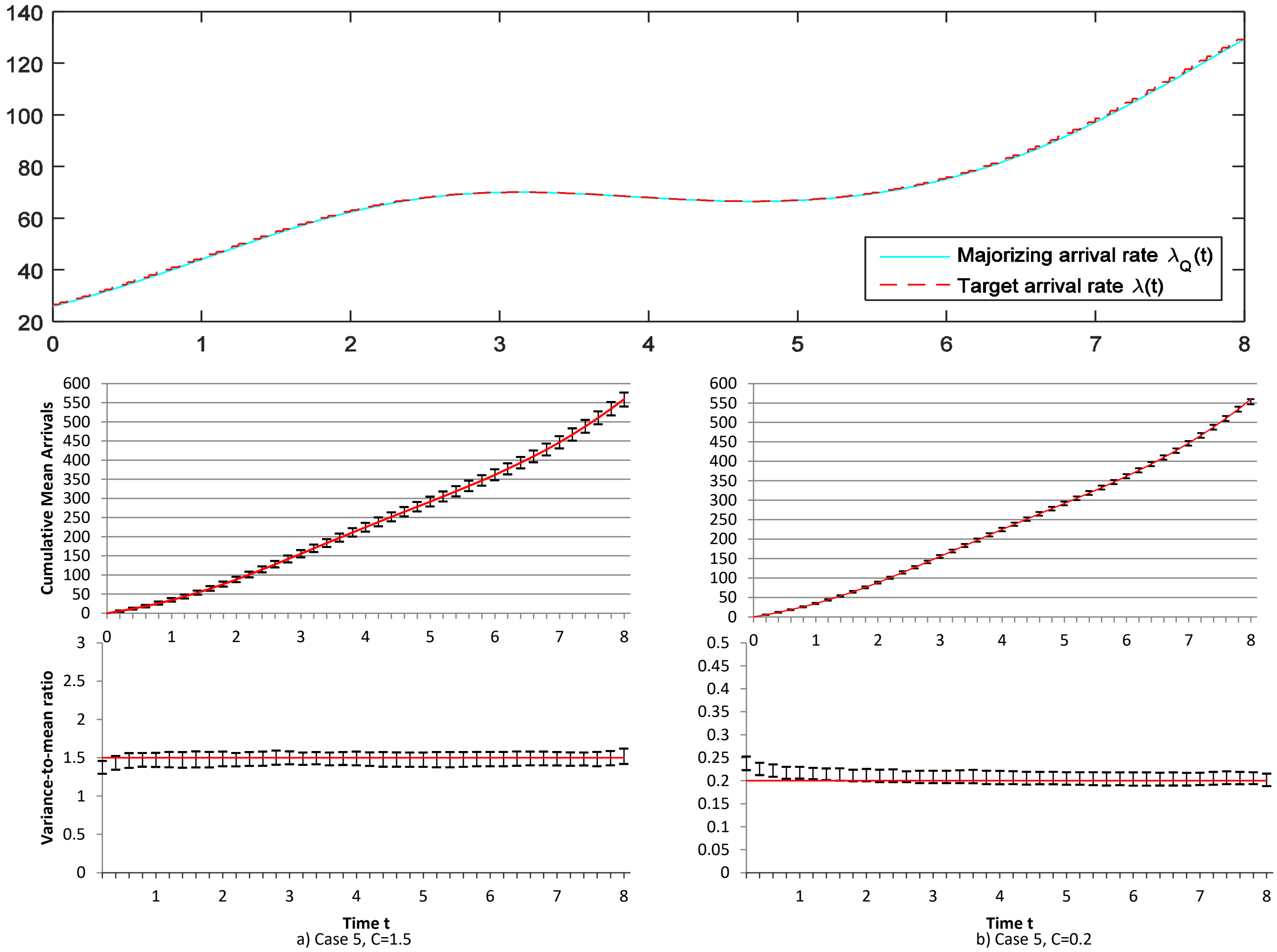

Figure S6 **CIATA-Ph Performance for Case-5 Arrival Rates and Dispersion Ratio $C = 1.5$ and 0.2: (i) (Top) 95% CIs for the Mean-Value Function $\mu(t)$ with $C = 1.5$ (Left) and $C = 0.2$ (Right); and (ii) (Bottom) 95% CIs for $C_Q(t)$ with $C = 1.5$ (Left) and $C = 0.2$ (Right) .**

Table S4 **CIATA-Ph–based Closeness Measures for Case 5.**

Case 1				
C	$\Delta_{\mathbb{T}}(\widehat{\mu}_Q)$	$\Delta^*_{\mathbb{T}}(\widehat{\mu}_Q)$	$\Delta_{\mathbb{T}}(\widehat{C}_Q)$	$\Delta^*_{\mathbb{T}}(\widehat{C}_Q)$
0.2	$0.050\% \pm 0.037\%$	1.072%	$4.476\% \pm 0.773\%$	18.743%
1.5	$0.043\% \pm 0.011\%$	0.313%	$1.515\% \pm 0.295\%$	8.475%

S3.3 Warm-up Times and Accuracy of CIATA-Ph

In this section we evaluate the performance of CIATA-Ph by examining (i) the speed of convergence of CIATA-Ph and (ii) the closeness of the estimated values $\widehat{\mu}_Q(t)$ and $\widehat{C}_Q(t)$ to their respective theoretical counterparts $\mu(t)$ and C for various cases and for various scenarios. The experimental results show that both the speed of convergence and the quality of the estimated values varies among the instances of each test process (case) and across the test processes. The speed of convergence depends on the properties of the rate function $\lambda(t)$ and on the dispersion ratio C, while the quality of estimated values depends on Q, the number of subintervals used to construct the step rate function $\widetilde{\lambda}_Q(t)$ that majorizes and closely approximates $\lambda(t)$.

To evaluate the speed of convergence of CIATA-Ph, we define the *warm-up time* (WUT) t_w of a test process as the minimum time value such that C lies in the associated confidence interval when $t \geq t_w$. The WUT indicates the speed of convergence of CIATA-Ph for each test process. We compare the WUT t_w of all the three cases and all the four scenarios in Table S5.

As shown in Table S5, for every case and for every scenario the associated CI includes the corresponding value of C when $t \geq 1.8$. This indicates a fast convergence of CIATA-Ph for all the cases and for all the scenarios. In all the three cases, when the desired asymptotic dispersion ratio $C = 1.5$, 0.8, or 0.2, the value of C falls in the corresponding confidence interval when $t \geq 0.2$. It follows that the speed of convergence of CIATA-Ph is relatively faster when $C = 1.5$, 0.8, and 0.2.

To examine the closeness of the estimated values $\widehat{\mu}_Q(t)$ and $\widehat{C}_Q(t)$ to the respective values $\mu(t)$ and C, we continue to use the APD $\Delta_{\mathbb{T}}(\cdot)$ and MPD $\Delta^*_{\mathbb{T}}(\cdot)$ defined in (17). Specifically, we computed $\Delta_{\mathbb{T}}(\widehat{\mu}_Q)$, $\Delta^*_{\mathbb{T}}(\widehat{\mu}_Q)$, $\Delta_{\mathbb{T}}(\widehat{C}_Q)$, and $\Delta^*_{\mathbb{T}}(\widehat{C}_Q)$ and the approximate 95% CIs for $\mathrm{E}[\Delta_{\mathbb{T}}(\widehat{\mu}_Q)]$ and $\mathrm{E}[\Delta_{\mathbb{T}}(\widehat{C}_Q)]$ for every case and for every scenario. The results for each scenario in Case 1 are reported in Table S6. We obtained similar results for Cases 2 and 3.

Table S5 **Warm-Up Time of All the Cases and All the Scenarios Generated by CIATA-Ph Algorithm.**

case number	$C = 10$	$C = 1.5$	$C = 0.8$	$C = 0.2$
Case 1	1.2	0.2	0.2	0.2
Case 2	1.2	0.2	0.2	0.2
Case 3	1.8	0.2	0.2	0.2

As observed in Table S6, for all the scenarios the average percentage discrepancy of $\widehat{\mu}_Q$ is less than 1.5% while the maximum percentage discrepancy of $\widehat{\mu}_Q$ is less than 3%. This indicates that the estimated value of $\mu(t)$ is relatively close to the true value for all the scenarios. When $C \leq 1$, we obtained significantly smaller values of the measures $\Delta_{\mathbb{T}}(\widehat{C}_Q)$ and $\Delta^*_{\mathbb{T}}(\widehat{C}_Q)$ for the scenario $C = 0.8$ than those for $C = 0.2$. When $C \geq 1$, the values of these two measures for the scenario $C = 1.5$ are much smaller than those for the scenarios $C = 10$, 15, and 20. In general when C is close to 1, CIATA-Ph can generate NNPPs with estimated values of C relatively close to the corresponding true values.

Table S6 Closeness Measures for Case 1, Generated by CIATA-Ph Algorithm.

Scenario	$\Delta_{\mathbb{T}}(\widehat{\mu}_Q)$	$\Delta^*_{\mathbb{T}}(\widehat{\mu}_Q)$	$\Delta_{\mathbb{T}}(\widehat{C}_Q)$	$\Delta^*_{\mathbb{T}}(\widehat{C}_Q)$
$C = 20$	1.164%± 0.133%	2.499%	14.082%± 2.108%	58.960%
$C = 15$	1.409% ± 0.261%	2.966%	11.536%± 1.694%	50.735%
$C = 10$	0.957% ± 0.19%	2.287%	10.324% ± 1.146%	38.490%
$C = 1.5$	0.819% ± 0.176%	2.463%	1.901%± 0.35%	5.233%
$C = 0.8$	0.915%±0.200%	2.028%	1.669%±0.323%	5.163%
$C = 0.2$	0.918%±0.197%	2.538%	3.530% ±0.266%	6.150%

The magnitudes of the four measures $\Delta_{\mathbb{T}}(\widehat{\mu}_Q)$, $\Delta^*_{\mathbb{T}}(\widehat{\mu}_Q)$, $\Delta_{\mathbb{T}}(\widehat{C}_Q)$, and $\Delta^*_{\mathbb{T}}(\widehat{C}_Q)$ vary significantly among the test processes. This variation indicates that the adequacy of a CIATA-Ph–generated NNPP depends on the characteristics of the given rate and mean-value functions as well as the given dispersion ratio.

Finally, supplementing the good performance of CIATA-Ph as shown in Figures 2, 3, S2, S3, S4 and S5, we provide the corresponding detailed values of the CI estimators $\mu(t)$ and $C_Q(t)$ for experiments with arrival rates in all three cases and $C = 10, 1.5, 0.8$ and 0.2, in Tables S7–S30.

Table S7 CIATA-Ph–Generated 95% CI Estimators for $\mu(t),\ t \in (0,8]$, in Case 1 with $C = 1.5$.

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{\mu}(t)_{0.975}$	14.074	28.956	46.220	51.764	54.327	66.425	80.827	97.565	103.036	105.600
$\widehat{\mu}(t)$	11.590	25.283	41.401	46.650	49.106	60.646	74.440	90.594	95.875	98.342
$\widehat{\mu}(t)_{0.025}$	9.106	21.609	36.582	41.536	43.886	54.867	68.053	83.622	88.714	91.084
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{\mu}(t)_{0.975}$	117.647	131.843	148.543	153.964	156.475	168.426	182.593	199.091	204.415	206.884
$\widehat{\mu}(t)$	109.956	123.735	139.866	145.108	147.557	159.098	172.802	188.866	194.052	196.474
$\widehat{\mu}(t)_{0.025}$	102.265	115.627	131.189	136.253	138.639	149.770	163.011	178.642	183.689	186.064
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{\mu}(t)_{0.975}$	218.770	232.872	249.373	254.744	257.222	269.255	283.286	299.629	304.969	307.465
$\widehat{\mu}(t)$	208.070	221.814	237.885	243.111	245.537	257.240	270.976	286.966	292.247	294.680
$\widehat{\mu}(t)_{0.025}$	197.370	210.756	226.397	231.478	233.853	245.226	258.665	274.303	279.524	281.894
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{\mu}(t)_{0.975}$	319.273	333.213	349.590	354.833	357.283	369.026	382.999	399.478	404.861	406.866
$\widehat{\mu}(t)$	306.213	319.865	335.862	341.046	343.448	354.966	368.723	384.874	390.152	392.134
$\widehat{\mu}(t)_{0.025}$	293.152	306.517	322.134	327.258	329.612	340.906	354.448	370.270	375.443	377.401

Table S8 **CIATA-Ph–Generated 95% CI Estimators for $C_Q(t)$, $t \in (0,8]$, in Case 1 with $C = 1.5$.**

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{C}(t)_{0.975}$	1.565	1.557	1.613	1.613	1.598	1.576	1.573	1.532	1.527	1.530
$\widehat{C}(t)$	1.422	1.422	1.486	1.490	1.480	1.472	1.459	1.424	1.423	1.426
$\widehat{C}(t)_{0.025}$	1.278	1.288	1.360	1.367	1.363	1.368	1.345	1.316	1.319	1.322
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{C}(t)_{0.975}$	1.563	1.555	1.574	1.573	1.568	1.595	1.610	1.606	1.614	1.608
$\widehat{C}(t)$	1.442	1.423	1.443	1.452	1.448	1.470	1.486	1.485	1.488	1.483
$\widehat{C}(t)_{0.025}$	1.320	1.291	1.312	1.331	1.328	1.345	1.363	1.363	1.362	1.358
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{C}(t)_{0.975}$	1.610	1.618	1.631	1.629	1.625	1.635	1.634	1.616	1.603	1.605
$\widehat{C}(t)$	1.477	1.477	1.486	1.489	1.487	1.500	1.491	1.489	1.478	1.480
$\widehat{C}(t)_{0.025}$	1.344	1.336	1.340	1.349	1.350	1.365	1.348	1.361	1.353	1.355
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{C}(t)_{0.975}$	1.618	1.610	1.621	1.614	1.618	1.628	1.616	1.609	1.610	1.603
$\widehat{C}(t)$	1.489	1.488	1.498	1.488	1.488	1.493	1.483	1.484	1.486	1.483
$\widehat{C}(t)_{0.025}$	1.361	1.366	1.376	1.362	1.359	1.358	1.350	1.359	1.363	1.363

Table S9 **CIATA-Ph–Generated 95% CI Estimators for $\mu(t)$, $t \in (0,8]$, in Case 1 with $C = 0.2$.**

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{\mu}(t)_{0.975}$	12.610	26.796	43.354	48.762	51.261	63.089	77.061	93.411	98.769	101.283
$\widehat{\mu}(t)$	11.510	25.264	41.392	46.613	49.043	60.599	74.349	90.446	95.697	98.144
$\widehat{\mu}(t)_{0.025}$	10.410	23.732	39.430	44.465	46.824	58.109	71.636	87.482	92.626	95.005
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{\mu}(t)_{0.975}$	113.036	126.961	143.265	148.599	151.064	162.725	176.625	192.917	198.223	200.694
$\widehat{\mu}(t)$	109.701	123.437	139.516	144.748	147.175	158.701	172.476	188.573	193.800	196.228
$\widehat{\mu}(t)_{0.025}$	106.366	119.913	135.767	140.898	143.285	154.676	168.327	184.228	189.376	191.762
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{\mu}(t)_{0.975}$	212.414	226.332	242.557	247.898	250.346	262.045	275.904	292.088	297.388	299.842
$\widehat{\mu}(t)$	207.828	221.595	237.678	242.924	245.346	256.909	270.670	286.688	291.902	294.334
$\widehat{\mu}(t)_{0.025}$	203.243	216.859	232.800	237.950	240.345	251.772	265.436	281.287	286.417	288.826
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{\mu}(t)_{0.975}$	311.467	325.351	341.506	346.754	349.218	360.853	374.720	390.906	396.178	398.154
$\widehat{\mu}(t)$	305.856	319.660	335.694	340.882	343.321	354.870	368.614	384.709	389.957	391.902
$\widehat{\mu}(t)_{0.025}$	300.244	313.969	329.882	335.010	337.425	348.886	362.507	378.511	383.736	385.649

Table S10 **CIATA-Ph–Generated 95% CI Estimators for $C_Q(t)$, $t \in (0,8]$, in Case 1 with $C = 0.2$.**

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{C}(t)_{0.975}$	0.229	0.221	0.217	0.219	0.217	0.221	0.219	0.218	0.220	0.222
$\widehat{C}(t)$	0.209	0.202	0.202	0.204	0.203	0.206	0.206	0.205	0.205	0.207
$\widehat{C}(t)_{0.025}$	0.189	0.184	0.187	0.189	0.188	0.191	0.192	0.191	0.190	0.192
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{C}(t)_{0.975}$	0.222	0.224	0.228	0.229	0.231	0.228	0.226	0.227	0.227	0.228
$\widehat{C}(t)$	0.208	0.208	0.210	0.211	0.212	0.209	0.207	0.209	0.209	0.210
$\widehat{C}(t)_{0.025}$	0.193	0.192	0.192	0.193	0.194	0.191	0.188	0.191	0.191	0.191
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{C}(t)_{0.975}$	0.227	0.227	0.224	0.225	0.226	0.228	0.226	0.228	0.228	0.228
$\widehat{C}(t)$	0.208	0.209	0.207	0.208	0.208	0.209	0.207	0.209	0.210	0.209
$\widehat{C}(t)_{0.025}$	0.189	0.191	0.189	0.190	0.190	0.190	0.188	0.190	0.192	0.191
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{C}(t)_{0.975}$	0.226	0.221	0.222	0.222	0.223	0.223	0.226	0.226	0.224	0.224
$\widehat{C}(t)$	0.208	0.204	0.204	0.205	0.205	0.206	0.208	0.208	0.206	0.206
$\widehat{C}(t)_{0.025}$	0.190	0.187	0.186	0.187	0.187	0.189	0.190	0.189	0.188	0.187

Table S11 **CIATA-Ph–Generated 95% CI Estimators for $\mu(t)$, $t \in (0,8]$, in Case 1 with $C = 10$.**

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{\mu}(t)_{0.975}$	16.742	34.274	53.370	59.306	61.967	74.963	90.254	108.034	113.689	116.347
$\widehat{\mu}(t)$	11.540	25.523	41.651	46.859	49.217	60.809	74.392	90.463	95.581	98.029
$\widehat{\mu}(t)_{0.025}$	6.338	16.771	29.931	34.411	36.467	46.655	58.531	72.893	77.474	79.710
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{\mu}(t)_{0.975}$	128.869	143.641	161.349	166.940	169.507	181.801	196.687	213.740	219.328	221.927
$\widehat{\mu}(t)$	109.474	123.070	139.351	144.500	146.913	158.356	172.151	188.003	193.284	195.755
$\widehat{\mu}(t)_{0.025}$	90.079	102.499	117.352	122.061	124.318	134.911	147.616	162.266	167.239	169.583
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{\mu}(t)_{0.975}$	234.089	249.029	266.300	271.935	274.556	286.737	301.248	318.131	323.590	326.110
$\widehat{\mu}(t)$	207.279	221.231	237.412	242.727	245.228	256.730	270.446	286.436	291.652	294.062
$\widehat{\mu}(t)_{0.025}$	180.469	193.433	208.523	213.518	215.900	226.724	239.644	254.742	259.714	262.013
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{\mu}(t)_{0.975}$	338.220	352.666	369.392	374.985	377.526	389.655	404.081	420.898	426.302	428.442
$\widehat{\mu}(t)$	305.601	319.323	335.172	340.440	342.869	354.390	368.169	384.157	389.383	391.422
$\widehat{\mu}(t)_{0.025}$	272.982	285.981	300.953	305.895	308.211	319.125	332.256	347.415	352.463	354.403

Table S12 CIATA-Ph–Generated 95% CI Estimators for $C_Q(t)$, $t \in (0,8]$, in Case 1 with $C = 10$.

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{C}(t)_{0.975}$	7.144	8.827	9.675	9.834	9.883	10.063	10.225	10.267	10.391	10.402
$\widehat{C}(t)$	6.630	8.202	8.989	9.130	9.165	9.281	9.434	9.467	9.560	9.575
$\widehat{C}(t)_{0.025}$	6.116	7.577	8.304	8.426	8.446	8.499	8.643	8.666	8.729	8.748
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{C}(t)_{0.975}$	10.504	10.382	10.514	10.619	10.573	10.671	10.795	10.786	10.754	10.713
$\widehat{C}(t)$	9.688	9.660	9.721	9.796	9.775	9.813	9.871	9.907	9.887	9.867
$\widehat{C}(t)_{0.025}$	8.871	8.938	8.928	8.972	8.977	8.954	8.947	9.029	9.020	9.020
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{C}(t)_{0.975}$	10.679	10.727	10.806	10.785	10.778	10.809	10.819	10.844	10.835	10.830
$\widehat{C}(t)$	9.821	9.876	9.934	9.935	9.929	9.951	9.932	9.908	9.893	9.885
$\widehat{C}(t)_{0.025}$	8.963	9.025	9.062	9.084	9.080	9.093	9.045	8.972	8.951	8.940
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{C}(t)_{0.975}$	10.823	10.802	10.777	10.815	10.813	10.830	10.797	10.829	10.835	10.841
$\widehat{C}(t)$	9.881	9.871	9.874	9.915	9.912	9.953	9.921	9.936	9.904	9.908
$\widehat{C}(t)_{0.025}$	8.939	8.940	8.970	9.014	9.010	9.076	9.045	9.044	8.973	8.974

Table S13 CIATA-Ph-Generated 95% CI Estimators for $\mu(t)$, $t \in (0,8]$, in Case 1 with $C = 0.8$.

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{\mu}(t)_{0.975}$	13.451	28.144	44.930	50.367	52.925	64.913	79.161	95.773	101.138	103.637
$\widehat{\mu}(t)$	11.571	25.348	41.363	46.581	49.028	60.575	74.348	90.404	95.624	98.048
$\widehat{\mu}(t)_{0.025}$	9.690	22.551	37.796	42.796	45.132	56.237	69.535	85.034	90.109	92.459
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{\mu}(t)_{0.975}$	115.521	129.546	146.039	151.345	153.853	165.720	179.811	196.199	201.497	203.987
$\widehat{\mu}(t)$	109.582	123.269	139.373	144.578	147.028	158.625	172.425	188.531	193.750	196.191
$\widehat{\mu}(t)_{0.025}$	103.642	116.993	132.707	137.811	140.204	151.531	165.038	180.863	186.002	188.394
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{\mu}(t)_{0.975}$	215.783	229.746	246.255	251.557	254.033	265.844	279.870	296.293	301.521	304.001
$\widehat{\mu}(t)$	207.717	221.478	237.683	242.886	245.322	256.952	270.765	286.890	292.047	294.494
$\widehat{\mu}(t)_{0.025}$	199.650	213.210	229.111	234.215	236.611	248.059	261.660	277.488	282.573	284.988
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{\mu}(t)_{0.975}$	315.674	329.669	345.980	351.276	353.723	365.430	379.404	395.863	401.191	403.169
$\widehat{\mu}(t)$	306.024	319.809	335.895	341.086	343.518	355.025	368.764	384.978	390.213	392.176
$\widehat{\mu}(t)_{0.025}$	296.374	309.949	325.810	330.895	333.314	344.620	358.123	374.092	379.234	381.184

Table S14 CIATA-Ph–Generated 95% CI Estimators for $C_Q(t)$, $t \in (0,8]$, in Case 1 with $C = 0.8$.

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{C}(t)_{0.975}$	0.860	0.865	0.864	0.866	0.867	0.863	0.871	0.892	0.893	0.891
$\widehat{C}(t)$	0.798	0.806	0.804	0.804	0.809	0.811	0.814	0.833	0.831	0.832
$\widehat{C}(t)_{0.025}$	0.736	0.747	0.743	0.742	0.750	0.759	0.757	0.774	0.769	0.773
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{C}(t)_{0.975}$	0.908	0.906	0.910	0.903	0.904	0.904	0.899	0.893	0.882	0.880
$\widehat{C}(t)$	0.841	0.836	0.834	0.829	0.829	0.830	0.828	0.816	0.811	0.810
$\widehat{C}(t)_{0.025}$	0.774	0.765	0.758	0.754	0.753	0.756	0.756	0.739	0.739	0.740
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{C}(t)_{0.975}$	0.893	0.879	0.881	0.879	0.881	0.878	0.877	0.879	0.879	0.879
$\widehat{C}(t)$	0.820	0.808	0.809	0.810	0.809	0.805	0.801	0.806	0.804	0.803
$\widehat{C}(t)_{0.025}$	0.746	0.736	0.737	0.740	0.738	0.732	0.726	0.734	0.730	0.728
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{C}(t)_{0.975}$	0.871	0.867	0.860	0.864	0.859	0.866	0.871	0.868	0.874	0.873
$\widehat{C}(t)$	0.797	0.795	0.792	0.796	0.793	0.798	0.803	0.804	0.808	0.806
$\widehat{C}(t)_{0.025}$	0.722	0.724	0.724	0.728	0.726	0.729	0.734	0.741	0.741	0.738

Table S15 CIATA-Ph–Generated 95% CI Estimators for $\mu(t)$, $t \in (0,8]$, in Case 2 with $C = 1.5$.

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{\mu}(t)_{0.975}$	14.110	29.635	47.699	53.607	56.400	69.950	86.481	106.023	112.341	115.410
$\widehat{\mu}(t)$	11.606	25.824	42.767	48.360	51.001	63.883	79.706	98.497	104.592	107.550
$\widehat{\mu}(t)_{0.025}$	9.103	22.014	37.835	43.113	45.601	57.816	72.931	90.971	96.844	99.689
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{\mu}(t)_{0.975}$	130.222	148.072	169.379	176.416	179.779	195.900	215.428	238.909	246.660	250.318
$\widehat{\mu}(t)$	121.830	139.131	159.770	166.617	169.891	185.605	204.756	227.555	235.141	238.712
$\widehat{\mu}(t)_{0.025}$	113.438	130.191	150.161	156.818	160.003	175.310	194.083	216.201	223.621	227.105
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{\mu}(t)_{0.975}$	268.257	290.014	315.661	324.248	328.310	347.816	371.687	400.173	409.691	414.246
$\widehat{\mu}(t)$	256.246	277.500	302.674	311.060	315.057	334.259	357.629	385.481	394.780	399.256
$\widehat{\mu}(t)_{0.025}$	244.234	264.985	289.688	297.872	301.805	320.702	343.571	370.788	379.869	384.265
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{\mu}(t)_{0.975}$	436.043	462.513	493.896	504.274	509.227	533.130	562.176	597.083	608.483	612.923
$\widehat{\mu}(t)$	420.629	446.578	477.496	487.696	492.567	516.070	544.659	579.017	590.264	594.583
$\widehat{\mu}(t)_{0.025}$	405.215	430.644	461.095	471.119	475.907	499.009	527.142	560.951	572.046	576.244

Table S16 CIATA-Ph–Generated 95% CI Estimators for $C_Q(t)$, $t \in (0,8]$, in Case 2 with $C = 1.5$.

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{C}(t)_{0.975}$	1.521	1.609	1.621	1.618	1.619	1.623	1.604	1.629	1.635	1.634
$\widehat{C}(t)$	1.411	1.472	1.488	1.489	1.495	1.506	1.504	1.504	1.502	1.503
$\widehat{C}(t)_{0.025}$	1.302	1.334	1.355	1.360	1.371	1.388	1.403	1.379	1.369	1.373
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{C}(t)_{0.975}$	1.642	1.639	1.646	1.637	1.634	1.614	1.577	1.606	1.594	1.591
$\widehat{C}(t)$	1.512	1.503	1.512	1.508	1.505	1.493	1.455	1.481	1.475	1.475
$\widehat{C}(t)_{0.025}$	1.383	1.368	1.378	1.378	1.377	1.372	1.333	1.356	1.356	1.358
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{C}(t)_{0.975}$	1.608	1.613	1.579	1.582	1.573	1.544	1.552	1.582	1.595	1.596
$\widehat{C}(t)$	1.473	1.477	1.457	1.462	1.457	1.436	1.443	1.464	1.472	1.471
$\widehat{C}(t)_{0.025}$	1.339	1.341	1.334	1.341	1.341	1.329	1.335	1.345	1.349	1.347
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{C}(t)_{0.975}$	1.595	1.607	1.595	1.592	1.594	1.592	1.588	1.595	1.586	1.596
$\widehat{C}(t)$	1.476	1.486	1.473	1.473	1.473	1.474	1.472	1.474	1.469	1.478
$\widehat{C}(t)_{0.025}$	1.357	1.365	1.351	1.354	1.352	1.356	1.357	1.352	1.353	1.361

Table S17 CIATA-Ph–Generated 95% CI Estimators for $\mu(t)$, $t \in (0,8]$, in Case 2 with $C = 0.2$.

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{\mu}(t)_{0.975}$	13.612	28.736	46.507	52.331	55.117	68.620	84.697	103.873	110.296	113.323
$\widehat{\mu}(t)$	11.687	25.886	42.868	48.443	51.101	64.051	79.643	98.279	104.514	107.466
$\widehat{\mu}(t)_{0.025}$	9.761	23.035	39.229	44.554	47.084	59.481	74.588	92.685	98.732	101.609
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{\mu}(t)_{0.975}$	128.056	145.854	166.983	173.961	177.365	193.473	213.128	236.303	243.958	247.608
$\widehat{\mu}(t)$	121.795	139.135	159.801	166.614	169.925	185.722	204.985	227.742	235.262	238.842
$\widehat{\mu}(t)_{0.025}$	115.533	132.417	152.619	159.267	162.485	177.970	196.841	219.180	226.566	230.076
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{\mu}(t)_{0.975}$	265.388	286.819	312.343	320.781	324.815	344.547	368.303	396.509	405.946	410.409
$\widehat{\mu}(t)$	256.256	277.365	302.546	310.880	314.864	334.218	357.592	385.403	394.746	399.142
$\widehat{\mu}(t)_{0.025}$	247.124	267.910	292.748	300.979	304.914	323.889	346.882	374.297	383.545	387.874
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{\mu}(t)_{0.975}$	431.996	458.289	489.655	499.994	504.947	528.769	557.613	592.279	603.721	608.019
$\widehat{\mu}(t)$	420.472	446.403	477.305	487.573	492.471	516.024	544.609	578.864	590.180	594.447
$\widehat{\mu}(t)_{0.025}$	408.947	434.516	464.954	475.152	479.996	503.278	531.605	565.449	576.638	580.875

Table S18 CIATA-Ph–Generated 95% CI Estimators for $C_Q(t)$, $t \in (0,8]$, in Case 2 with $C = 0.2$.

time t	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
$\widehat{C}(t)_{0.975}$	0.238	0.234	0.238	0.236	0.236	0.236	0.236	0.230	0.229	0.229
$\widehat{C}(t)$	0.219	0.214	0.219	0.216	0.216	0.218	0.218	0.213	0.212	0.213
$\widehat{C}(t)_{0.025}$	0.199	0.194	0.200	0.196	0.197	0.199	0.200	0.197	0.196	0.196
time t	2.2	2.4	2.6	2.8	3	3.2	3.4	3.6	3.8	4
$\widehat{C}(t)_{0.975}$	0.227	0.225	0.222	0.222	0.220	0.219	0.218	0.219	0.219	0.218
$\widehat{C}(t)$	0.210	0.207	0.204	0.204	0.203	0.201	0.201	0.202	0.202	0.201
$\widehat{C}(t)_{0.025}$	0.193	0.189	0.185	0.186	0.185	0.184	0.184	0.185	0.186	0.183
time t	4.2	4.4	4.6	4.8	5	5.2	5.4	5.6	5.8	6
$\widehat{C}(t)_{0.975}$	0.216	0.216	0.217	0.218	0.218	0.220	0.223	0.223	0.225	0.225
$\widehat{C}(t)$	0.199	0.200	0.200	0.201	0.201	0.203	0.205	0.205	0.206	0.207
$\widehat{C}(t)_{0.025}$	0.183	0.184	0.182	0.184	0.183	0.186	0.187	0.186	0.187	0.188
time t	6.2	6.4	6.6	6.8	7	7.2	7.4	7.6	7.8	8
$\widehat{C}(t)_{0.975}$	0.226	0.223	0.222	0.221	0.223	0.225	0.224	0.220	0.221	0.221
$\widehat{C}(t)$	0.207	0.205	0.205	0.204	0.205	0.207	0.208	0.204	0.205	0.205
$\widehat{C}(t)_{0.025}$	0.189	0.188	0.188	0.187	0.188	0.190	0.191	0.189	0.189	0.189

Table S19 CIATA-Ph-Generated 95% CI Estimators for $\mu(t)$, $t \in (0,8]$, in Case 2 with $C = 10$.

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{\mu}(t)_{0.975}$	16.774	34.543	54.466	60.680	63.673	78.023	95.741	116.452	123.099	126.226
$\widehat{\mu}(t)$	11.614	25.687	42.631	48.134	50.805	63.562	79.209	97.940	104.053	106.975
$\widehat{\mu}(t)_{0.025}$	6.455	16.831	30.795	35.588	37.937	49.101	62.678	79.429	85.007	87.724
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{\mu}(t)_{0.975}$	141.529	160.126	182.400	189.662	193.171	210.278	230.740	254.956	262.718	266.373
$\widehat{\mu}(t)$	121.149	138.263	158.993	165.751	169.033	185.039	204.315	227.030	234.460	237.964
$\widehat{\mu}(t)_{0.025}$	100.770	116.401	135.586	141.839	144.894	159.800	177.890	199.104	206.203	209.555
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{\mu}(t)_{0.975}$	284.510	306.780	333.435	342.138	346.299	366.549	391.077	420.224	430.060	434.650
$\widehat{\mu}(t)$	255.126	276.103	301.232	309.495	313.484	332.816	356.206	384.117	393.492	397.929
$\widehat{\mu}(t)_{0.025}$	225.743	245.426	269.029	276.853	280.668	299.084	321.335	348.009	356.923	361.207
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{\mu}(t)_{0.975}$	456.797	483.805	515.955	526.569	531.658	556.192	585.622	620.728	632.373	637.009
$\widehat{\mu}(t)$	419.214	445.004	475.750	485.990	490.870	514.392	542.852	576.531	587.714	592.164
$\widehat{\mu}(t)_{0.025}$	381.631	406.203	435.545	445.410	450.082	472.592	500.082	532.335	543.056	547.319

Table S20 CIATA-Ph-Generated 95% CI Estimators for $C_Q(t)$, $t \in (0,8]$, in Case 2 with $C = 10$.

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{C}(t)_{0.975}$	6.890	8.953	9.616	9.682	9.739	10.053	10.404	10.469	10.452	10.404
$\widehat{C}(t)$	6.470	8.317	8.919	9.001	9.046	9.305	9.646	9.715	9.719	9.689
$\widehat{C}(t)_{0.025}$	6.049	7.681	8.223	8.320	8.353	8.557	8.888	8.960	8.985	8.975
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{C}(t)_{0.975}$	10.351	10.451	10.551	10.622	10.619	10.706	10.590	10.604	10.495	10.480
$\widehat{C}(t)$	9.654	9.694	9.648	9.687	9.694	9.736	9.632	9.638	9.581	9.563
$\widehat{C}(t)_{0.025}$	8.957	8.936	8.744	8.752	8.770	8.766	8.674	8.673	8.667	8.646
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{C}(t)_{0.975}$	10.453	10.483	10.646	10.679	10.683	10.606	10.527	10.534	10.508	10.497
$\widehat{C}(t)$	9.586	9.622	9.699	9.728	9.719	9.703	9.649	9.585	9.605	9.599
$\widehat{C}(t)_{0.025}$	8.718	8.761	8.751	8.777	8.754	8.801	8.772	8.637	8.701	8.701
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{C}(t)_{0.975}$	10.414	10.411	10.427	10.381	10.379	10.394	10.365	10.389	10.471	10.488
$\widehat{C}(t)$	9.561	9.567	9.585	9.569	9.577	9.606	9.515	9.530	9.566	9.586
$\widehat{C}(t)_{0.025}$	8.707	8.722	8.744	8.756	8.776	8.818	8.665	8.671	8.661	8.684

Table S21 CIATA-Ph–Generated 95% CI Estimators for $\mu(t)$, $t \in (0,8]$, in Case 2 with $C = 0.8$.

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{\mu}(t)_{0.975}$	13.612	28.736	46.507	52.331	55.117	68.620	84.697	103.873	110.296	113.323
$\widehat{\mu}(t)$	11.687	25.886	42.868	48.443	51.101	64.051	79.643	98.279	104.514	107.466
$\widehat{\mu}(t)_{0.025}$	9.761	23.035	39.229	44.554	47.084	59.481	74.588	92.685	98.732	101.609
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{\mu}(t)_{0.975}$	128.056	145.854	166.983	173.961	177.365	193.473	213.128	236.303	243.958	247.608
$\widehat{\mu}(t)$	121.795	139.135	159.801	166.614	169.925	185.722	204.985	227.742	235.262	238.842
$\widehat{\mu}(t)_{0.025}$	115.533	132.417	152.619	159.267	162.485	177.970	196.841	219.180	226.566	230.076
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{\mu}(t)_{0.975}$	265.388	286.819	312.343	320.781	324.815	344.547	368.303	396.509	405.946	410.409
$\widehat{\mu}(t)$	256.256	277.365	302.546	310.880	314.864	334.218	357.592	385.403	394.746	399.142
$\widehat{\mu}(t)_{0.025}$	247.124	267.910	292.748	300.979	304.914	323.889	346.882	374.297	383.545	387.874
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{\mu}(t)_{0.975}$	431.996	458.289	489.655	499.994	504.947	528.769	557.613	592.279	603.721	608.019
$\widehat{\mu}(t)$	420.472	446.403	477.305	487.573	492.471	516.024	544.609	578.864	590.180	594.447
$\widehat{\mu}(t)_{0.025}$	408.947	434.516	464.954	475.152	479.996	503.278	531.605	565.449	576.638	580.875

Table S22 **CIATA-Ph–Generated 95% CI Estimators for $C_Q(t)$, $t \in (0,8]$, in Case 2 with $C = 0.8$.**

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{C}(t)_{0.025}$	0.863	0.838	0.854	0.866	0.891	0.878	0.890	0.891	0.879	0.882
$\widehat{C}(t)$	0.806	0.775	0.793	0.807	0.828	0.814	0.826	0.818	0.813	0.813
$\widehat{C}(t)_{0.975}$	0.749	0.713	0.732	0.747	0.765	0.750	0.763	0.744	0.747	0.744

time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{C}(t)_{0.025}$	0.885	0.886	0.878	0.881	0.882	0.892	0.895	0.894	0.901	0.905
$\widehat{C}(t)$	0.822	0.822	0.816	0.817	0.819	0.822	0.821	0.819	0.822	0.822
$\widehat{C}(t)_{0.975}$	0.758	0.758	0.754	0.753	0.755	0.753	0.748	0.745	0.742	0.739

time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{C}(t)_{0.025}$	0.908	0.903	0.889	0.893	0.891	0.893	0.898	0.895	0.891	0.892
$\widehat{C}(t)$	0.824	0.818	0.812	0.816	0.815	0.818	0.820	0.821	0.818	0.818
$\widehat{C}(t)_{0.975}$	0.739	0.732	0.735	0.738	0.740	0.742	0.743	0.747	0.745	0.744

time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{C}(t)_{0.025}$	0.886	0.882	0.875	0.876	0.880	0.874	0.876	0.879	0.880	0.886
$\widehat{C}(t)$	0.815	0.812	0.807	0.811	0.812	0.807	0.805	0.803	0.804	0.808
$\widehat{C}(t)_{0.975}$	0.745	0.742	0.739	0.745	0.744	0.739	0.733	0.727	0.727	0.730

Table S23 **CIATA-Ph–Generated 95% CI Estimators for $\mu(t)$, $t \in (0,8]$, in Case 3 with C=1.5.**

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{\mu}(t)_{0.975}$	13.947	28.592	45.127	50.281	52.643	63.801	77.101	92.238	97.176	99.399
$\widehat{\mu}(t)$	11.453	24.897	40.419	45.306	47.565	58.196	70.848	85.356	90.113	92.260
$\widehat{\mu}(t)_{0.025}$	8.959	21.202	35.710	40.330	42.487	52.592	64.595	78.475	83.050	85.120

time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{\mu}(t)_{0.975}$	110.008	122.498	137.119	141.823	144.008	154.603	167.264	182.268	187.110	189.378
$\widehat{\mu}(t)$	102.476	114.497	128.642	133.221	135.366	145.600	157.918	172.471	177.208	179.459
$\widehat{\mu}(t)_{0.025}$	94.944	106.497	120.165	124.619	126.723	136.596	148.571	162.674	167.306	169.541

time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{\mu}(t)_{0.975}$	200.331	213.509	229.322	234.588	237.064	249.094	263.743	281.057	286.812	289.652
$\widehat{\mu}(t)$	190.105	202.938	218.344	223.469	225.890	237.618	251.825	268.765	274.379	277.145
$\widehat{\mu}(t)_{0.025}$	179.880	192.368	207.366	212.351	214.717	226.143	239.907	256.473	261.946	264.639

time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{\mu}(t)_{0.975}$	303.216	319.734	339.469	346.188	349.391	365.426	385.213	409.104	417.110	420.289
$\widehat{\mu}(t)$	290.426	306.621	326.036	332.610	335.754	351.470	370.752	394.189	402.039	405.147
$\widehat{\mu}(t)_{0.025}$	277.636	293.508	312.602	319.031	322.117	337.513	356.291	379.273	386.969	390.005

Table S24 CIATA-Ph–Generated 95% CI Estimators for $C_Q(t)$, $t \in (0,8]$, in Case 3 with $C = 1.5$.

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{C}(t)_{0.975}$	1.611	1.600	1.620	1.610	1.588	1.583	1.615	1.625	1.630	1.635
$\widehat{C}(t)$	1.470	1.462	1.460	1.461	1.448	1.442	1.469	1.472	1.475	1.482
$\widehat{C}(t)_{0.025}$	1.328	1.323	1.300	1.313	1.308	1.300	1.323	1.318	1.320	1.328
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{C}(t)_{0.975}$	1.646	1.655	1.637	1.628	1.621	1.644	1.632	1.633	1.630	1.614
$\widehat{C}(t)$	1.493	1.504	1.499	1.491	1.482	1.502	1.488	1.492	1.488	1.475
$\widehat{C}(t)_{0.025}$	1.341	1.353	1.361	1.353	1.344	1.360	1.343	1.351	1.346	1.336
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{C}(t)_{0.975}$	1.622	1.621	1.619	1.620	1.620	1.625	1.655	1.653	1.648	1.655
$\widehat{C}(t)$	1.481	1.482	1.484	1.488	1.486	1.492	1.514	1.508	1.511	1.515
$\widehat{C}(t)_{0.025}$	1.340	1.344	1.349	1.356	1.352	1.358	1.372	1.362	1.373	1.375
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{C}(t)_{0.975}$	1.645	1.646	1.608	1.608	1.606	1.612	1.632	1.628	1.638	1.646
$\widehat{C}(t)$	1.511	1.502	1.477	1.480	1.479	1.480	1.504	1.503	1.508	1.513
$\widehat{C}(t)_{0.025}$	1.377	1.358	1.346	1.352	1.352	1.349	1.377	1.378	1.378	1.379

Table S25 CIATA-Ph–Generated 95% CI Estimators for $\mu(t),\ t \in (0,8]$, in Case 3 with $C = 0.2$.

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{\mu}(t)_{0.975}$	13.382	27.596	43.798	48.980	51.324	62.382	75.319	90.299	95.175	97.433
$\widehat{\mu}(t)$	11.465	24.831	40.285	45.227	47.496	58.177	70.688	85.163	89.896	92.073
$\widehat{\mu}(t)_{0.025}$	9.548	22.067	36.771	41.473	43.668	53.971	66.056	80.027	84.617	86.713

time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{\mu}(t)_{0.975}$	107.903	120.415	134.969	139.702	141.922	152.421	164.972	179.887	184.780	187.043
$\widehat{\mu}(t)$	102.252	114.393	128.598	133.224	135.401	145.684	157.957	172.496	177.278	179.498
$\widehat{\mu}(t)_{0.025}$	96.602	108.371	122.226	126.746	128.880	138.946	150.941	165.106	169.776	171.953

time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{\mu}(t)_{0.975}$	197.999	211.286	226.949	232.075	234.527	246.526	260.970	278.334	284.132	286.948
$\widehat{\mu}(t)$	190.253	203.212	218.541	223.585	225.975	237.748	251.987	269.046	274.710	277.451
$\widehat{\mu}(t)_{0.025}$	182.507	195.137	210.133	215.095	217.423	228.970	243.004	259.758	265.287	267.955

time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{\mu}(t)_{0.975}$	300.346	316.836	336.510	343.170	346.378	362.275	381.775	405.567	413.574	416.688
$\widehat{\mu}(t)$	290.616	306.884	326.292	332.868	336.018	351.696	370.943	394.422	402.279	405.345
$\widehat{\mu}(t)_{0.025}$	280.887	296.932	316.073	322.566	325.659	341.117	360.111	383.277	390.985	394.002

Table S26 CIATA-Ph–Generated 95% CI Estimators for $C_Q(t)$, $t \in (0,8]$, in Case 3, $C = 0.2$.

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{C}(t)_{0.975}$	0.227	0.216	0.213	0.217	0.216	0.215	0.214	0.217	0.219	0.218
$\widehat{C}(t)$	0.210	0.200	0.196	0.197	0.197	0.197	0.197	0.200	0.202	0.201
$\widehat{C}(t)_{0.025}$	0.193	0.183	0.180	0.177	0.178	0.179	0.179	0.183	0.185	0.184

time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{C}(t)_{0.975}$	0.219	0.219	0.222	0.221	0.221	0.221	0.217	0.218	0.218	0.217
$\widehat{C}(t)$	0.201	0.201	0.205	0.203	0.203	0.203	0.199	0.200	0.199	0.199
$\widehat{C}(t)_{0.025}$	0.183	0.183	0.187	0.185	0.186	0.185	0.182	0.182	0.181	0.181

time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{C}(t)_{0.975}$	0.216	0.216	0.215	0.215	0.215	0.214	0.213	0.214	0.216	0.216
$\widehat{C}(t)$	0.198	0.198	0.197	0.198	0.198	0.197	0.196	0.196	0.198	0.198
$\widehat{C}(t)_{0.025}$	0.180	0.180	0.179	0.181	0.181	0.181	0.179	0.178	0.180	0.180

time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{C}(t)_{0.975}$	0.215	0.215	0.213	0.212	0.211	0.211	0.213	0.217	0.219	0.219
$\widehat{C}(t)$	0.197	0.197	0.196	0.196	0.196	0.194	0.196	0.198	0.200	0.199
$\widehat{C}(t)_{0.025}$	0.179	0.180	0.180	0.180	0.180	0.178	0.178	0.180	0.180	0.180

Table S27 CIATA-Ph–Generated 95% CI Estimators for $\mu(t),\ t \in (0,8]$, in Case 3 with $C = 10$.

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{\mu}(t)_{0.975}$	16.742	33.611	51.555	57.124	59.674	71.595	85.644	101.964	107.358	109.846
$\widehat{\mu}(t)$	11.279	24.595	39.670	44.507	46.720	57.243	69.675	84.558	89.256	91.440
$\widehat{\mu}(t)_{0.025}$	5.815	15.578	27.785	31.889	33.765	42.891	53.706	67.153	71.154	73.033
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{\mu}(t)_{0.975}$	120.387	133.420	148.545	153.777	156.256	168.313	181.433	196.789	202.011	204.366
$\widehat{\mu}(t)$	101.592	113.845	128.194	132.788	134.958	145.209	157.481	171.947	176.765	178.989
$\widehat{\mu}(t)_{0.025}$	82.796	94.269	107.843	111.799	113.660	122.106	133.528	147.106	151.518	153.612
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{\mu}(t)_{0.975}$	215.982	230.177	246.692	252.120	254.655	267.142	282.185	299.866	306.254	309.527
$\widehat{\mu}(t)$	189.783	202.800	218.146	223.171	225.609	237.247	251.298	268.380	274.151	276.966
$\widehat{\mu}(t)_{0.025}$	163.583	175.422	189.600	194.221	196.562	207.353	220.410	236.893	242.048	244.405
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{\mu}(t)_{0.975}$	323.819	340.153	360.654	367.542	371.069	388.371	409.483	432.988	440.947	444.309
$\widehat{\mu}(t)$	290.228	306.139	325.246	331.701	334.800	350.446	369.664	392.680	400.474	403.784
$\widehat{\mu}(t)_{0.025}$	256.637	272.125	289.838	295.859	298.531	312.520	329.846	352.372	360.000	363.258

Table S28 CIATA-Ph–Generated 95% CI Estimators for $C_Q(t),\ t \in (0,8]$, in Case 3 with $C = 10$.

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{C}(t)_{0.975}$	6.799	8.759	9.462	9.652	9.680	9.719	9.787	9.995	10.064	10.089
$\widehat{C}(t)$	6.4741	8.311	9.017	9.193	9.217	9.292	9.328	9.386	9.409	9.422
$\widehat{C}(t)_{0.025}$	6.148	7.862	8.572	8.734	8.753	8.864	8.868	8.778	8.754	8.754
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{C}(t)_{0.975}$	10.134	10.178	10.173	10.180	10.195	10.213	10.266	10.391	10.441	10.461
$\widehat{C}(t)$	9.435	9.492	9.511	9.551	9.580	9.609	9.646	9.769	9.816	9.829
$\widehat{C}(t)_{0.025}$	8.735	8.806	8.850	8.922	8.965	9.006	9.027	9.146	9.191	9.197
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{C}(t)_{0.975}$	10.597	10.658	10.829	10.822	10.793	10.830	10.902	10.916	10.958	10.937
$\widehat{C}(t)$	9.910	9.979	10.087	10.049	10.023	10.043	10.189	10.149	10.196	10.194
$\widehat{C}(t)_{0.025}$	9.223	9.300	9.344	9.276	9.253	9.256	9.477	9.382	9.433	9.451
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{C}(t)_{0.975}$	10.913	10.966	11.008	11.034	11.072	11.100	11.009	11.039	10.989	10.944
$\widehat{C}(t)$	10.199	10.229	10.288	10.324	10.366	10.431	10.405	10.399	10.362	10.319
$\widehat{C}(t)_{0.025}$	9.485	9.493	9.567	9.615	9.660	9.762	9.800	9.760	9.7345	9.694

Table S29 CIATA-Ph–Generated 95% CI Estimators for $\mu(t),\ t \in (0,8]$, in Case 3 with $C = 0.8$.

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{\mu}(t)_{0.975}$	13.382	27.596	43.798	48.980	51.324	62.382	75.319	90.299	95.175	97.433
$\widehat{\mu}(t)$	11.465	24.831	40.285	45.227	47.496	58.177	70.688	85.163	89.896	92.073
$\widehat{\mu}(t)_{0.025}$	9.548	22.067	36.771	41.473	43.668	53.971	66.056	80.027	84.617	86.713
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{\mu}(t)_{0.975}$	107.903	120.415	134.969	139.702	141.922	152.421	164.972	179.887	184.780	187.043
$\widehat{\mu}(t)$	102.252	114.393	128.598	133.224	135.401	145.684	157.957	172.496	177.278	179.498
$\widehat{\mu}(t)_{0.025}$	96.602	108.371	122.226	126.746	128.880	138.946	150.941	165.106	169.776	171.953
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{\mu}(t)_{0.975}$	197.999	211.286	226.949	232.075	234.527	246.526	260.970	278.334	284.132	286.948
$\widehat{\mu}(t)$	190.253	203.212	218.541	223.585	225.975	237.748	251.987	269.046	274.710	277.451
$\widehat{\mu}(t)_{0.025}$	182.507	195.137	210.133	215.095	217.423	228.970	243.004	259.758	265.287	267.955
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{\mu}(t)_{0.975}$	300.346	316.836	336.510	343.170	346.378	362.275	381.775	405.567	413.574	416.688
$\widehat{\mu}(t)$	290.616	306.884	326.292	332.868	336.018	351.696	370.943	394.422	402.279	405.345
$\widehat{\mu}(t)_{0.025}$	280.887	296.932	316.073	322.566	325.659	341.117	360.111	383.277	390.985	394.002

Table S30 CIATA-Ph–Generated 95% CI Estimators for $C_Q(t)$, $t \in (0,8]$, in Case 3 with $C = 0.8$.

time t	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$\widehat{C}(t)_{0.975}$	0.872	0.870	0.859	0.866	0.879	0.899	0.882	0.874	0.880	0.874
$\widehat{C}(t)$	0.827	0.819	0.806	0.815	0.824	0.850	0.837	0.830	0.834	0.832
$\widehat{C}(t)_{0.025}$	0.783	0.768	0.753	0.764	0.769	0.801	0.791	0.786	0.789	0.791
time t	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$\widehat{C}(t)_{0.975}$	0.879	0.886	0.887	0.884	0.887	0.884	0.890	0.885	0.886	0.890
$\widehat{C}(t)$	0.839	0.846	0.842	0.845	0.849	0.843	0.844	0.839	0.838	0.839
$\widehat{C}(t)_{0.025}$	0.800	0.806	0.797	0.805	0.811	0.802	0.797	0.794	0.790	0.789
time t	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$\widehat{C}(t)_{0.975}$	0.894	0.884	0.875	0.868	0.866	0.882	0.884	0.884	0.878	0.879
$\widehat{C}(t)$	0.849	0.840	0.828	0.822	0.820	0.833	0.837	0.835	0.829	0.830
$\widehat{C}(t)_{0.025}$	0.803	0.797	0.780	0.777	0.774	0.784	0.789	0.785	0.780	0.781
time t	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
$\widehat{C}(t)_{0.975}$	0.872	0.871	0.880	0.873	0.873	0.864	0.847	0.844	0.846	0.842
$\widehat{C}(t)$	0.824	0.825	0.833	0.825	0.824	0.821	0.809	0.810	0.810	0.808
$\widehat{C}(t)_{0.025}$	0.775	0.779	0.787	0.778	0.775	0.778	0.771	0.776	0.774	0.773