INTRODUCTION TO LÉVY PROCESSES

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INTRODUCTION

Lévy processes, now widely used to construct and analyze financial models, were named in honor of the French mathematician *Paul Lévy* (1886–1971). As one of the founders of modern probability theory, Lévy made major contributions to the study of Gaussian processes, stable laws, infinitely divisible distributions, and processes with independent and stationary increments, which is now known as *Lévy processes*.

Lévy processes provide ingredients for building a rich class of continuous-time stochastic processes, such as *Poisson processes* and *Brownian motions*, which are two fundamental examples. For more details, see [1] and Chapter 5 of [2]. The applications of Lévy processes have immensely been emerging in many areas, for instance, queuing theory and financial engineering.

In this review article, we intend to give an introductory lecture and provide fundamental results on Lévy processes. In the section titled "Preliminary," we state preliminaries on probability theory and stochastic processes; in the section titled "Lévy Processes," we formally introduce Lévy processes and their distributional properties; in the section titled "Examples of Lévy Processes," we illustrate five important examples of Lévy processes; in the section titled "Poisson Random Measures," we present *Poisson random measures* (PRMs), the building blocks of the pure-jump part of Lévy processes; in the sections titled "Lévy–Itô Decomposition" and "Lévy–Khintchine Formula," we introduce the two most important theorems: *Lévy–Itô decomposition* and *Lévy–Khintchine formula*; and finally in the section titled "Path Properties," we introduce four kinds of Lévy processes distinguished by different path properties.

PRELIMINARY

Probability Space and Random Variables

Let the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ denote a *probability space*, where $\mathcal{F}$ is the σ-field (-algebra) of the underlying *sample space* Ω, and $\mathbb{P} : \mathcal{F} \to [0,1]$ is the *probability measure* associated. A *random variable* is a real-valued $\mathcal{F}$-measurable function $X{:}\Omega \to \mathbb{R}$, where $\mathcal{F}$-measurability means that for any set B in the Borel σ-field $\mathcal{B}$, we have $X^{-1}(B) \in \mathcal{F}$. Note that the random variable X induces a probability measure μ_X on $\mathbb{R}$, defined by $\mu_X(B) \equiv \mathbb{P}(X^{-1}(B)) \equiv \mathbb{P}(\{\omega \in \Omega : X(\omega) \in B\})$ for any Borel set B in $\mathcal{B}$.

Let $F_X(\cdot) \equiv \mathbb{P}(X \le \cdot)$ and $F_X^c \equiv 1 - F_X$ be the *cumulative distribution function* (*CDF*) and *complement cumulative distribution function* (*CCDF*) of a random variable X. Two random variables X and Y are said to be *equal in distribution*, denoted by $X \stackrel{\mathcal{D}}{=} Y$, if $F_X = F_Y$. Two random variables X and Y are said to be *independent*, denoted by $X \perp Y$, if $\mathbb{P}(X \le x, Y \le y) = \mathbb{P}(X \le x)\mathbb{P}(Y \le y)$ for all $x, y \in \mathbb{R}$. Given the independence, the distribution (induced measure) of $X + Y$ is given by convolution of measures, that is, $\mu_{X+Y} = \mu_X * \mu_Y$ $\mu_X * \mu_Y(A) \equiv \int_{\mathbb{R}} \mu_X(A - y)\mu_Y(\mathrm{d}y)$, $A \in \mathcal{B}$ and $\mu_Y(\mathrm{d}y) \equiv \mathbb{P}(\{\omega \in \Omega : y < Y(\omega) < y + \mathrm{d}y\})$.

Let $\mathbb{E}[X] \equiv \int_\Omega X \,\mathrm{d}\mathbb{P} = \int_{\mathbb{R}} x\, \mu_X(\mathrm{d}x) = \int_{\mathbb{R}} x\, \mathrm{d}F(x)$ be the *expectation* of X, provided that $\int_{\mathbb{R}} |x|\, dF(x) < \infty$. Note that, if two random variables X and Y are independent, we have $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

Characteristic Functions

The *characteristic function* (CF) of a random variable X is defined by $\Psi_X(\theta) \equiv \mathbb{E}[\mathrm{e}^{i\theta X}]$, with

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$\theta \in \mathbb{R}$ and $i \equiv \sqrt{-1}$. It has the following properties: (i) $\left|\Psi_X(\theta)\right| \leq 1$; (ii) $\Psi_X(\theta)$ is real-valued if and only if X is symmetric; (iii) $\Psi_X(\theta)$ is Hermitian, that is, $\Psi_X(-\theta) = \overline{\Psi_X(\theta)}$; and (iv) $\mathbb{E}\left[X^n\right] = i^{-n}\frac{d^n}{d\theta^n}\Psi_X(\theta)\Big|_{\theta=0}$. Note that the CF uniquely determines the distribution of a random variable.

The independence among the random variables can be understood through their CFs. Namely, the random variables $X_1, \ldots, X_n$ are independent if and only if

$$\mathbb{E}\left[e^{i\sum_{k=1}^{n}\theta_k X_k}\right] = \prod_{k=1}^{n}\Psi_{X_k}(\theta_k). \quad (1)$$

Counting Processes and Counting Measures

A *stochastic process* $\{X(t), t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a parameterized collection of random variables $\{X(t)\}_{t\geq 0}$, assuming values in $\mathbb{R}$, that is, $X : [0,\infty) \times \Omega \to \mathbb{R}$. For each fixed $\omega \in \Omega$, $\{X(t,\omega), t >= 0\}$ a sample path of the process, is a deterministic function and, for each fixed $t \geq 0$, $X(t,\cdot)$ is a random variable. A *counting process* $\{N(t), t \geq 0\}$ is a stochastic process that counts the total number of events occurred, and thus must be (i) nonnegative; (ii) integer-valued; and (iii) nondecreasing. Note that to understand (iii), we interpret $N(t) - N(s)$ as the total number of events occurred within time interval $(s, t]$.

A *counting measure* can be viewed as an induced measure by the associated counting process. Let $0 \leq T_1 \leq T_2 \leq \ldots$ be the sequence of the occurrence times of events. For a measurable set $A \subset [0,\infty)$, define the measure $M(\omega, A) \equiv \#\{i \geq 1, T_i(\omega) \in A\}$. Note that the measure $M(\omega,\cdot)$ depends on $\omega \in \Omega$; it is thus a *random measure*. More details on random measures will be discussed in the section titled "Poisson Random Measures."

LÉVY PROCESSES

A *random walk*, sum of independent and identically distributed (IID) random variables, provides the simplest example of discrete-time stochastic processes. We hereby introduce the definition of *Lévy* processes, the continuous-time analogs of random walks.

Definition 1. **[Lévy Processes].** A continuous-time process $\{X(t), t \geq 0\}$ is called a *Lévy process* if

(i) $X(0) = 0$;
(ii) it has *independent* increments, that is, $X(t_4) - X(t_3) \perp X(t_2) - X(t_1)$ for all $0 < t_1 < t_2 < t_3 < t_4$;
(iii) it has *stationary* increments, that is, $X(t_2) - X(t_1) \stackrel{\mathcal{D}}{=} X(t_2 - t_1)$ for all $0 < t_1 < t_2$;
(iv) its path is *stochastically continuous*, that is, $\lim_{s\to t}\mathbb{P}(|X(t) - X(s)| > \epsilon) = 0$, for $\epsilon > 0$.

Note that the *stochastic continuity* condition in property (iv) does not imply that sample paths of Lévy processes are continuous. For instance, *Poisson processes*, special cases of Lévy processes, are pure-jump processes. Property (iv) serves to rule out the processes with discontinuities at deterministic times (known as the "calender effect").

Infinite Divisibility

A distribution F of a random variable X is called *infinitely divisible* if there exists n IID random variables $Y_{n,1}, \ldots, Y_{n,n}$ such that the sum $\sum_{i=1}^{n} Y_{n,i}$ follows distribution F for all $n \geq 2$.

Famous examples of infinitely divisible distributions are *Poisson*, *Gaussian*, and *Gamma*. A random variable X following the above distributions can be expressed by a sum of n IID random variables following the same distribution but with modified parameter (dependent on n). For instance, a Gaussian random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ is equal in distribution to $\sum_{i=1}^{n} Y_i$, where $Y_1, \ldots, Y_n$ are IID following $\mathcal{N}(\mu/n, \sigma^2/n)$. Other examples are *Pareto*, *lognormal*, *Cauchy*, *stable*, and *student* distributions.

Distributional Properties of Lévy Processes

Lévy processes possess the *infinitely divisible* property. Consider a fixed time $t > 0$, we have

$$X(t) = \sum_{k=1}^{n} Y_k^{(t)}, \text{ where}$$

$$Y_k^{(t)} \equiv X\left(\frac{k\,t}{n}\right) - X\left(\frac{(k-1)t}{n}\right),$$

denotes the increment of the process in the interval $((k-1)t/n, kt/n]$, $1 \le k \le n$. The infinite divisibility of $X(t)$ easily follows by properties (ii) and (iii) of the definition of Lévy processes.

Following Equation (1) and the definition of infinite divisibility, the CF of $X(t)$ with $t = n$ can be written as

$$\Psi_{X(n)}(\theta) = \prod_{k=1}^{n} \Psi_{Y_k^{(n)}}(\theta) = (\Psi_{X(1)}(\theta))^n,$$

where the last equality holds as $Y_k^{(n)} \overset{\mathcal{D}}{=} Y_1^{(n)} = X(n/n) = X(1)$. We can then easily extend the above argument from integer-valued t to all $t > 0$. Therefore, for all $t > 0$, we can write the CF

$$\Psi_{X(t)}(\theta) = (\Psi_{X(1)}(\theta))^t \equiv \mathrm{e}^{-t\,\psi(\theta)},$$

where the function ψ is called the *characteristic exponent* (CE) of the Lévy process $\{X(t), t \ge 0\}$. As $\Psi_{X(1)}(\theta) = \mathrm{e}^{\psi(\theta)}$, the distribution (law) of the whole process is determined solely by the CE ψ, or equivalently by the distribution of the random variable $X(1)$. It is not hard to prove that for a distribution F that is infinitely divisible, there exists a Lévy process $\{X(t), t \ge 0\}$ such that the $X(1)$ follows F.

EXAMPLES OF LÉVY PROCESSES

We next introduce some concrete examples of Lévy processes. Some of these examples are fundamental building blocks to construct general Lévy processes.

Poisson Processes

A Lévy process $\{N(t), t \ge 0\}$ is a *Poisson process* with rate $\lambda > 0$ if, for a fixed t, the random variable $N(t)$ follows Poisson distribution with mean λt, that is, the *probability mass function* (PMF) $\mathbb{P}(N(t) = k) = \mathrm{e}^{-\lambda t}(\lambda t)^k/k!$, for $k = 0, 1, 2 \ldots$.

Poisson processes are pure-jump processes with positive unit jumps and are thus counting processes. The intertransition times between these jumps $T_1, T_2, \ldots$ form a sequence of IID *exponential* distribution with rate λ, having a *probability density function* (PDF) $f_{T_1}(x) = \lambda \mathrm{e}^{-\lambda t}\mathbf{1}_{\{t \ge 0\}}$. The CE of a Poisson process with rate λ is given by

$$\psi(\theta) = \lambda(1 - \mathrm{e}^{i\theta}). \tag{2}$$

Besides Lévy processes, Poisson processes are special cases of several other classical processes, such as *renewal processes* and *continuous-time Markov chains*.

Compound Poisson Processes

Given a Poisson process $\{N(t), t \ge 0\}$ and a sequence of IID random variables $\{\xi_i : i \ge 1\}$ that are independent with N, then the process $\{Y(t), t \ge 0\}$ with $Y(t) \equiv \sum_{i=1}^{N(t)} \xi_i$ is called a *compound Poisson process*. Compound Poisson processes generalize Poisson processes from unit jump sizes to general and random jump sizes. In general, compound Poisson processes are no longer counting processes because there may be negative jumps. Let G be the CDF of ξ_1, which describes the sizes of the jumps.

The fact that compound Poisson processes have stationary and independent increments simply follows from those properties of Poisson processes. The CE is given by

$$\psi(\theta) = \lambda \int_{\mathbb{R}} (1 - \mathrm{e}^{i\theta y})\mathrm{d}G(y). \tag{3}$$

Note that Equation (3) reduces to Equation (2) when the measure induced by ξ_1 degenerates to $\delta_1(\cdot)$, the dirac measure at point 1.

Brownian Motions

A Lévy process with continuous sample path (with probability 1) $\{W(t), t \ge 0\}$ is called a standard *Brownian motion* when, for a fixed t, the random variable $W(t) \sim \mathcal{N}(0, t)$, that is, $W(t)$ follows a Gaussian distribution with mean 0 and variance t. Let

$$X(t) \equiv a\,W(t) + b\,t. \tag{4}$$

Then $\{X(t), t \ge 0\}$ is called a *Brownian motion with drift*. It is obvious that $X(t) \sim \mathcal{N}(bt, a^2 t)$ with PDF

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi a^2 t}} \mathrm{e}^{-\frac{(x-bt)^2}{2a^2 t}}.$$

The CE of X is given by

$$\psi(\theta) = a^2\theta^2/2 - i\theta\, b. \tag{5}$$

Inverse Gaussian Processes

Regarding a standard Brownian motion with drift $\{X(t), t \geq 0\}$ defined in Equation (4) with $a = 1$ and $b > 0$, define the first passage time

$$\tau(t) \equiv \inf\{s > 0 : X(s) > t\}, \tag{6}$$

that is, the first time a Brownian motion crosses above the level $t \geq 0$. Then the process $\{\tau(t), t \geq 0\}$ is called an *inverse Gaussian process*.

We use the so-called *strong Markov* property of Brownian motions to show that $\{\tau(t), t \geq 0\}$ indeed is a Lévy process. Define $X^*(t) \equiv X(\tau(s) + t) - s$, then the new process $\{X^*(t), t \geq 0\}$, describing the evolution of the Brown motion after hitting level s, is equal in distribution to $\{X(t), t \geq 0\}$. The dynamics of X^* is also independent of $X(u)$ for $0 \leq u \leq \tau(s)$. For $0 \leq s < t$ let $\tau^*(t-s)$ be the first passage time (in the form of Equation (6)) of X^* at level $t - s$. We thus have that the remaining time to level t (given hitting s)

$$\tau(t) - \tau(s) = \tau^*(t-s) \stackrel{\mathcal{D}}{=} \tau(t-s)$$
$$\text{and} \quad \tau(t) - \tau(s) \perp \tau(s),$$

which prove that the inverse Gaussian process has stationary and independent increments.

The PDF of $\tau(t)$ is given by

$$f_{\tau(t)}(x) = \frac{t}{\sqrt{2\pi x^3}} \mathrm{e}^{tb} \mathrm{e}^{-\frac{1}{2}(t^2x^{-1} + b^2x)},$$

and the CE takes the form

$$\psi(\theta) = \sqrt{-2i\theta + b^2} - b.$$

Furthermore, the inverse Gaussian process is an example of a *subordinator*, a subclass of Lévy processes having nondecreasing paths (see section titled "Path Properties").

Stable Processes

A random variable Y is said to follow a *stable* distribution if, for all $n \geq 1$,

$$n^{1/\alpha} Y + b_n \stackrel{\mathcal{D}}{=} \sum_{i=1}^{n} Y_i, \tag{7}$$

where $Y_1, \ldots, Y_n$ are IID copies of Y, $\alpha \in (0, 2]$ and $b_n \in \mathbb{R}$. We say the distribution is *strictly stable* if $b_n = 0$. By subtracting b_n/n from each term of the right-hand side of Equation (7) and divide both sides by $n^{1/\alpha}$, one can easily verify that stable distributions are infinitely divisible. *Stable processes* thus form one class of Lévy processes whose CEs correspond to those of stable distributions.

We point out that the case $\alpha = 2$ corresponds to zero mean Gaussian random variables. For $\alpha \in (0, 2)$, CE has the form

$$\psi(\theta) \equiv c|\theta|^\alpha \Gamma(\theta, \alpha, \beta) + i\theta\eta, \quad \text{where}$$
$$\Gamma(\theta, \alpha, \beta) \equiv \begin{cases} 1 - i\beta \tan(\frac{\pi\alpha}{2})\mathrm{sgn}(\theta), & \text{if } \alpha \neq 1, \\ 1 + i\beta \frac{2}{\pi} \mathrm{sgn}(\theta) \log|\theta|, & \text{if } \alpha = 1, \end{cases}$$

where $-1 \leq \beta \leq 1$, $c > 0$, $\eta \in \mathbb{R}$ and the sign function $\mathrm{sgn}(x) \equiv \mathbf{1}_{\{x>0\}} - \mathbf{1}_{\{x<0\}}$.

POISSON RANDOM MEASURES

Because Poisson and compound Poisson processes play important roles in the construction of Lévy processes, we next introduce the concept PRM that is exploited to study the jump structure of Lévy processes.

Definition 2. **[Poisson Random Measure].** Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $E \subset \mathbb{R}^d$. Given a Radon measure μ (i.e., $\mu(A) < \infty$ for any compact set A). A PRM on E with intensity measure μ is an integer-valued random measure such that

(i) for $\omega \in \Omega$, $M(\omega, \cdot)$ is an integer-valued Radon measure on E;

(ii) for each measurable set $A \subset E$, $M(A) \equiv M(\cdot, A)$ is a Poisson random variable with mean $\mu(A)$;

(iii) for disjoint measurable sets $A_1, \ldots, A_n$, $M(A_1), \ldots, M(A_n)$ are independent Poisson random variables.

We can construct a PRM with any given μ. Assume $\mu(E) < \infty$ without loss of generality, we (i) first generate IID random variables $X_1, X_2, \ldots$ with $\mathbb{P}(X_i \in A) = \frac{\mu(A)}{\mu(E)}$; (ii) second generate an independent Poisson random variable $M(E)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with mean $\mu(E)$; (iii) and finally let $M(A) \equiv \sum_{i=1}^{M(E)} \mathbf{1}_{\{X_i \in A\}}$.

Constructing Jump Processes Using PRM

PRMs can be employed to construct jump processes. Consider $E = [0, T] \times \mathbb{R}\backslash\{0\}$, we can write

$$M = \sum_{n \geq 1} \delta_{(T_n, \xi_n)}, \quad \text{or} \quad M(E) = \sum_{n \geq 1} \mathbf{1}_{\{(T_n, \xi_n) \in E\}},$$

where $(T_n)_{n \geq 1}$ is an increasing sequence. Each point $[T_n(\omega), \xi_n(\omega)] \in [0, T] \times \mathbb{R}\backslash\{0\}$ corresponds to jump with size $\xi_n(\omega)$ at time $T_n(\omega)$. The second expression counts the total number of jumps in $[0, T]$. We hereby exclude 0 so that every jump has a nonzero size.

By integrating a measurable function f with respect to the PRM M, we can construct a jump process

$$\begin{aligned} X(t) &= \int_0^t \int_{\mathbb{R}\backslash\{0\}} f(s, y) M(\mathrm{d}s \times \mathrm{d}y) \\ &= \sum_{T_n \in [0,t]} f(T_n, \xi_n), \end{aligned}$$

The above jump process becomes the compound Poisson process introduced in the section titled "Examples of Lévy Processes" when $f(s, y) = y$ and $\mu(\mathrm{d}s \times \mathrm{d}y) = \lambda\, \mu_{\xi_1}(\mathrm{d}y)\mathrm{d}s$, where μ_{ξ_1} is the measure induced by ξ_1. It becomes a Poisson process when $\mu_{\xi_1} = \delta_1$, see section titled "Examples of Lévy Processes."

LÉVY–ITÔ DECOMPOSITION

We are now ready to present the most fundamental result of Lévy processes, the *Lévy–Itô decomposition*. The Lévy–Itô decomposition enables us to understand the path structures of a general Lévy process by decomposing the Lévy process into three independent auxiliary Lévy processes.

Consider a Lévy process $\{X(t), t \geq 0\}$. We let $\Delta X(t) \equiv X(t+) - X(t-)$ be the size of a jump, if any, at time t and let A be a measurable set in $\mathbb{R}$. A *jump measure* is defined by

$$M_X(B) \equiv \#\{[t, \Delta X(t)] \in B\}, \tag{8}$$

for a measurable set $B \subset [0, \infty) \times \mathbb{R}$. Note that $M_X([t_1, t_2] \times A)$ counts the number of jumps of the Lévy process X in $[t_1, t_2]$ with jump sizes in A. The *Lévy measure* associated with X is defined as

$$\nu_X(A) \equiv \mathbb{E}\left[\sum_{0 \leq t \leq 1} \mathbf{1}_{\{\Delta X(t) \neq 0, \Delta X(t) \in A\}}\right], \tag{9}$$

which describes the expected number of jumps with sizes in A, per unit time. Obviously, we have

$$\nu_X(A) = \mathbb{E}[M_X([0, 1] \times A)]. \tag{10}$$

Lévy–Itô Decomposition

A Lévy process $\{X(t), t \geq 0\}$ can be written as

$$X(t) = X^c(t) + X^l(t) + X^s(t), \tag{11}$$

where $X^c(t)$, $X^l(t)$, and $X^s(t)$ are three independent processes; $X^c(t) \equiv bt + a\, W(t)$ is a Brownian motion with drift as defined in Equation (4); the two terms $X^l(t)$ and $X^s(t)$ are defined by

$$X^l(t) \equiv \int_0^t \int_{|x| \geq 1} x\, M_X(\mathrm{d}u \times \mathrm{d}x) \tag{12}$$

and

$$X^s(t) \equiv \lim_{\epsilon \to 0} \int_0^t \int_{\epsilon \leq |x| < 1} x \widetilde{M}_X(\mathrm{d}u \times \mathrm{d}x), \tag{13}$$

with $\widetilde{M}_X(\mathrm{d}u \times \mathrm{d}x) \equiv M_X(\mathrm{d}u \times \mathrm{d}x) - \nu_X(\mathrm{d}x)\mathrm{d}u$ being the centered (compensated) version of the jump measure M_X, as defined in

Equation (8); the Lévy measure ν_X, as defined in Equation (9), is a Radon measure on $\mathbb{R}\backslash\{0\}$ such that

$$\int (|y|^2 \wedge 1)\nu_X(\mathrm{d}y) < \infty; \tag{14}$$

and $a, b \in \mathbb{R}$ are constants.

To better understand this important result, we remark on the meanings of the three terms of the right-hand side in Equation (11). First, the Brownian motion with drift $X^c(t)$, described by the drift b and volatility term a, characterizes the continuous part of the path of a Lévy process.

The second term in Equation (11), as elaborated in Equation (12), is a compound Poisson process with a finite number of jumps with size >1. Note that the finiteness here is guaranteed by condition Eq. (14). Here the threshold separating big jumps and small jumps is set to be 1 as a convention. Changing the threshold from 1 to an arbitrary constant $c > 0$ results in a refinement of the drift term b. See [3] for alternative representations with a general c.

Finally, as the measure ν_X may have a singularity at 0, there can be infinitely many small jumps whose sum does not necessarily converge. This prevents us from letting ϵ go to 0 directly for a compound Poisson process with amplitude between ϵ and 1. The third term in Equation (11), as elaborated in Equation (13), thus has to be centered by the measure $\nu_X(\mathrm{d}x)\mathrm{d}u$ to obtain convergence.

In summary, the Lévy–Itô decomposition implies that an arbitrary Lévy process can be decomposed into the sum of three independent components: (i) a Brownian motion with drift, (ii) a compound Poisson process with finite and big jumps, and (iii) a discontinuous process with an infinite number of small jumps. The distribution of a Lévy process is thus characterized by three parameters a, b, and ν_X.

LÉVY–KHINTCHINE FORMULA

Using the Lévy–Itô decomposition, we can quickly obtain the second fundamental result: the *Lévy–Khinchine formula*.

Lévy–Khinchine Formula

Consider a Lévy process $\{X(t), t \geq 0\}$ with parameter (a, b, ν_X), its CE has the form

$$\psi(\theta) = \left(\frac{1}{2}a^2\theta^2 - i\,b\theta\right) + \int_{|x|\geq 1} (1 - \mathrm{e}^{i\,\theta x})\nu_X(\mathrm{d}x) + \int_{|x|<1} (1 - \mathrm{e}^{i\,\theta x} + i\,\theta x)\nu_X(\mathrm{d}x). \tag{15}$$

This result can be quickly proved using the Lévy–Itô decomposition and Equation (1). It can be easily seen from Equations (5) and (3) that the first two terms of the right-hand side of Equation (15) are the CE's of a Brownian motion with drift and a compound Poisson process with jump amplitudes greater than 1, corresponding to the first two terms of the right-hand side in Equation (11). In addition, the third term in Equation (15) is the CE of Equation (13).

PATH PROPERTIES

In the section titled "Lévy–Itô Decomposition," the distribution of a Lévy process $\{X(t), t \geq 0\}$ is characterized by its parameters a, b, and ν_X. We now introduce four typical kinds of Lévy processes distinguished by different path properties, reflected by different parameters.

Continuous Paths

As both X^l and X^s in Equation (11) represent jumps, a Lévy process having a continuous sample path must be a Brownian motion with drift. As a result, we must have only the first term in Equation (11) with $\nu_X = 0$ and Equation (15) thus degenerates to Equation (5).

Piecewise Constant Paths

On the contrary, as X^c in Equation (11) represents the continuous part, a pure-jump Lévy process (thus having piecewise constant paths) must be a compound Poisson process. Hence, the Lévy process involves only the second two terms in Equation (11) with $a = b = 0$ and $\nu_X(\mathbb{R}) < \infty$), see Ref. 4 for more details. Consequently, Equation (15) thus degenerates to Equation (3) with $\nu_X(\mathrm{d}x) = \lambda\,\mathrm{d}G(x)$.

Finite Variation

Consider an interval $[0,T]$, a function f is said to be of *finite variation* if, for each partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, the total variation thereof is finite, that is,

$$TV(f,[0,T]) \equiv \sup_{\{t_k\}_{k=1}^n} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| < \infty.$$

As Brownian motions have infinite variation over any finite interval [5], a Lévy process with finite variation must not involve the Brownian term W, that is, we have $a = 0$ in Equation (15). In addition, as compound Poisson processes [denoted by X^l in Equation (12)] always have finite variation, we impose extra conditions such that X^s has finite variation. We require $\int_{|x|\leq 1} |x|\,\nu_X(\mathrm{d}x) < \infty$. In this case, Equation (11) simplifies to

$$X(t) = \tilde{b}\,t + \int_0^t \int_{x\in\mathbb{R}} x\,M_X(\mathrm{d}u \times \mathrm{d}x),$$
$$\text{where} \quad \tilde{b} \equiv b - \int_{|x|<1} x\,\nu_X(\mathrm{d}x).$$

Subordinators

A Lévy process having nondecreasing paths over time is called a *subordinator*. In this case, the Brownian term is again not involved because the path of a Brownian motion is not monotone. Hence, $a = 0$. In addition, we require the Lévy process to have merely positive jumps of finite variation and positive drift, that is, $\nu_X((-\infty,0]) = 0$, $\int (x \wedge 1)\nu_X(\mathrm{d}x) < \infty$ and $b \geq 0$. See Ref. 3 for more discussion on subordinators.

FURTHER READING

More detailed contents on Lévy processes can be referred to the following books [3,4, 6–8]. Readers who are interested in infinite divisibility properties are referred to [9]. Regarding applications of Lévy processes in queuing theory, we refer to the book [10] and a recent survey on Lévy-driven queues [11].

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