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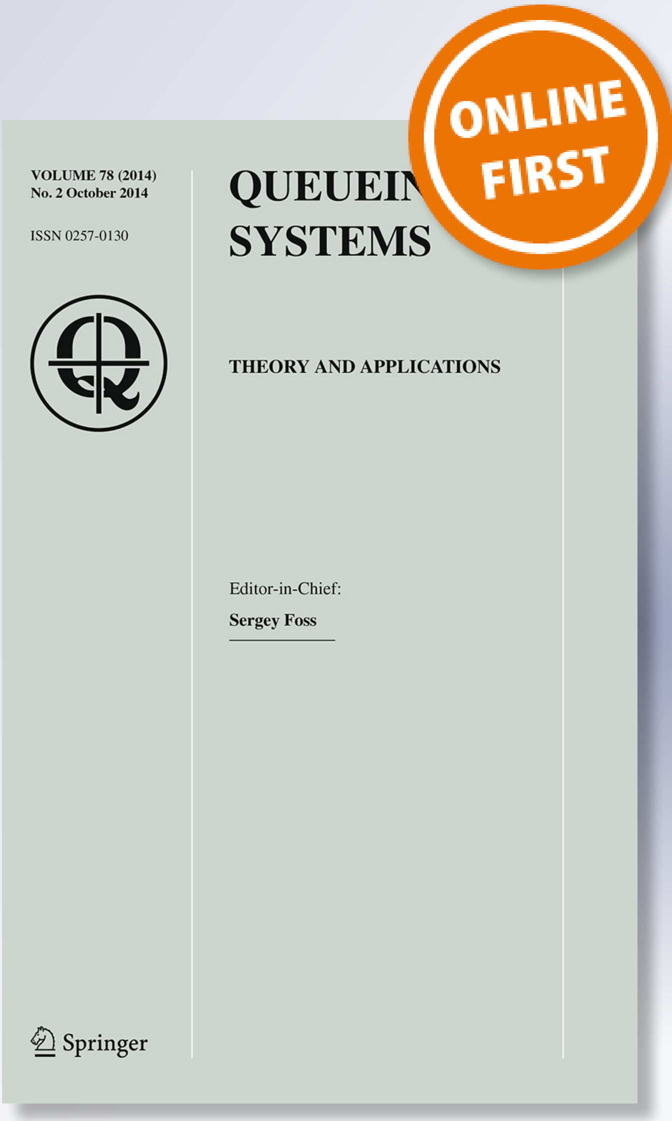
Queueing Systems
Theory and Applications

ISSN 0257-0130

Queueing Syst
DOI 10.1007/s11134-014-9419-5



Springer

Queueing Syst
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A law of iterated logarithm for multiclass queues with preemptive priority service discipline

Yongjiang Guo · Yunan Liu

Received: 31 July 2013 / Revised: 29 May 2014

Abstract A law of iterated logarithm (LIL) is established for a multiclass queueing model, having a preemptive priority service discipline, one server and K customer classes, with each class characterized by a renewal arrival process and i.i.d. service times. The LIL limits quantify the magnitude of asymptotic stochastic fluctuations of the stochastic processes compensated by their deterministic fluid limits. The LIL is established in three cases: underloaded, critically loaded, and overloaded, for five performance measures: queue length, workload, busy time, idle time, and number of departures. The proof of the LIL is based on a strong approximation approach, which approximates discrete performance processes with reflected Brownian motions. We conduct numerical examples to provide insights on these LIL results.

Keywords Law of iterated logarithm · Multiclass queues · Priority queues · Preemptive-resume discipline · Non-Markovian queues · Strong approximation

Mathematics Subject Classification 60K25 · 90B22 · 60F15 · 60J65

1 Introduction

In this paper, we develop a *law of iterated logarithm* (LIL) for the multiclass $(GI/GI)^K/1/\text{PPSD}$ queueing system, which has one server, K customer classes,

Y. Guo
School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China
e-mail: yongerguo@bupt.edu.cn

Y. Liu (✉)
Department of Industrial and Systems Engineering, North Carolina State University,
Raleigh, NC 27695-7906, USA
e-mail: yliu48@ncsu.edu

Published online: 12 September 2014

a *preemptive priority service discipline* (PPSD) with class i taking priority over class j for $1 \leq i < j \leq K$, class-dependent renewal arrival processes (the first GI) and *independent and identically distributed* (i.i.d.) non-exponential service times (the second GI).

Priority queueing systems In the literature of queueing theory, multiclass priority queues have largely captured researchers' attention because the models are relevant for many applications. For instance, in emergency rooms, patients are treated in the order based on their severity levels; in service systems such as call centers, VIP customers experience much less waiting time; in entertainment parks, customers who purchase "quickpasses" can jump over the long regular waiting lines. Various asymptotic theories have been developed for priority queueing models including heavy-traffic weak convergence results [1–3]; diffusion approximations [1,4,5]; and strong approximations (SAs) [6–9]. Because strong approximations are crucial building blocks for our proofs, we emphasize that among the literature on priority queues the most relevant work to the current paper is [7], which established the strong approximations for the $(GI/GI)^K/1$/PPSD model.

Law of iterated logarithm As a classical asymptotic result in probability theory, the LIL for a standard *Brownian motion* (BM) W is

$$\limsup_{T\to\infty} \frac{W(T)}{\sqrt{2T\log\log T}} = \limsup_{T\to\infty} \frac{|W(T)|}{\sqrt{2T\log\log T}} = 1 \tag{1}$$

$$= \limsup_{T\to\infty} \frac{\sup_{0\leq t\leq T} W(t)}{\sqrt{2T\log\log T}} = \limsup_{T\to\infty} \frac{\sup_{0\leq t\leq T} |W(t)|}{\sqrt{2T\log\log T}}, \quad \text{with probability } 1, \tag{2}$$

where (1) was the earliest LIL result developed by Lévy [10,11] and (2) was a later generalization by [12,13]. These versions of LIL in (1) and (2) are called the strong forms because they provide an explicit value (the "1" on the right-hand side of (1)) to quantify the asymptotic rate of the increasing variability for a standard BM. Motivated by the LIL for BM, various LIL results have later been developed for performance functions in queueing systems. Iglehart [14] developed LILs for the queue lengths of multiple channel queues; Sakalauskas and Minkevičius [15,16] obtained LILs for the queue lengths and waiting times of generalized Jackson networks assuming all queues are strictly overloaded. Also see [17] for LIL for strictly overloaded tandem queueing models. In contrast to the strong form in (1) and (2), Chen and Yao [18] provided a weak form of LIL for the queue length process Q (centered by its fluid function $\bar{Q}$) of the $GI/GI/1$ queue: they showed that $\sup_{0\leq t\leq T} \left|Q(t) - \bar{Q}(t)\right|$ is of the same order as the function $\sqrt{T\log\log T}$ as $T \to \infty$. The result is called the weak form because the LIL limit (as in (1)) was not clearly identified. Also see [7,18,19] for more results on weak LILs.

There also exists a body of literature on the *functional LIL* (FLIL). Analogous to (1), Strassen [20] developed the FLIL for the standard BM by considering a sequence of scaled BMs indexed by n, $W_n(t) \equiv W(nt)/\sqrt{n\log\log n}, n \geq 3$. Strassen showed that, with probability 1, the sequence $\{W_n, n \geq 3\}$ is relatively compact and that the limits of the convergent subsequences are contained in $\mathcal{K}_1$, which is the set of absolutely continuous functions x satisfying $x(0) = 0$ and $\int_0^1 \left[x'(t)\right]^2 \mathrm{d}t \le 1$. A major tool to establish FLIL results is the *continuous mapping theorem* (CMT); see Whitt [21].

Iglehart [14] adapted Strassen's approach to establish the FLIL for queue lengths, departures, and waiting times of the multiple channel queueing systems; also see Glynn and Whitt [22–24] for a Little's law version FLIL. In [3], Whitt hinted that the FLIL for the $(GI/GI)^K/1/\text{PPSD}$ could be developed using CMT, that is, the limits of convergent subsequences of performance functions could be characterized by some compact subset $\mathcal{K}_T$. However, we point out that (i) this set $\mathcal{K}_T$ usually does not have an explicit form so it is difficult to provide useful engineering approximations by directly using the FLIL results; (ii) in general, LIL results cannot simply be obtained as special cases of the corresponding FLILs (i.e., FLILs do not necessarily imply LILs); and (iii) the LILs are in some sense more difficult to establish because the powerful tool CMT cannot apply.

Our contributions We next summarize our contributions in four important directions. First, we establish a strong version of LILs in the form of (2) for all key performance functions of the $(GI/GI)^K/1/\text{PPSD}$ queueing system, including the queue length, workload (waiting time), idle time, busy time, and departure processes (see Sect. 2 for their definitions). Second, unlike many results in the literature which omit the difficult *critically loaded* (CL) and *underloaded* (UL) cases (thus only assuming the systems are strictly *overloaded* (OL)), we provide a complete analysis by covering all three regimes defined in terms of the traffic intensity ρ: (i) UL with $\rho < 1$, (ii) CL with $\rho = 1$, and (iii) OL with $\rho > 1$, see Sect. 2 for details of these three regimes. Third, we identify the LIL limits of the above performance measures as simple and analytic functions in terms of the model input parameters. Our results significantly refine the FLIL in [3,22–24] because these explicit limits can be exploited to provide useful engineering approximations for their corresponding stochastic processes. Fourth, our LIL limits provide interesting and sometimes counterintuitive observations. For instance, the LIL limits (in Theorems 2–6) are discontinuous in the traffic intensity ρ; in the OL case, in terms of the class index k, these limits always peak at the classes that deplete the remaining service resources; the LIL limits of high-priority classes are strongly influenced by their arrivals, while those of low-priority classes are independent with their arrivals. To elaborate and better understand these interesting observations, we provide comprehensive discussions (see Remarks 3–7) and concrete numerical examples (see Sect. 5).

A strong approximation approach The strong version of LIL for the $(GI/GI)^K/1/$ PPSD remained an open problem prior to the current paper because it is particularly difficult to deal with the CL and OL cases. To treat these cases, we follow three steps: The first is to relate the LILs of the performance functions to the LILs of their *strong approximations* (SAs). In order to establish convergence for the LILs, we next develop asymptotic theories for functions involving two BMs, see Lemmas 3 and 4 in Sect. 6. These results are legitimate in their own right and can be viewed the generalized version of the standard LIL of BMs in (1) and (2). Finally, we obtain the desired LIL limits by analyzing the (reflected) BMs given by the SAs.

Refining the *functional strong laws of large numbers* (FSLLNs) and fluid approximations that have been widely used to approximate the mean values of the corresponding stochastic processes, SAs provide effective estimates of the stochastic fluctuation around those mean values. We now demonstrate the idea of SA using a renewal process $\{N(t), t \geq 0\}$ with rate $\lambda > 0$ and interrenewal-time variance $\sigma^2 < \infty$. Let $\bar{N}(t) \equiv \lambda t$ and $\widetilde{N}$ be the SA for N. Suppose that the r-moment of the interrenewal-time exists with some $r > 2$. For a large $t > 0$, we write

$$N(t) \approx \widetilde{N}(t) \equiv \bar{N}(t) + \lambda^{3/2}\sigma W(t). \tag{3}$$

In addition, the error of the SA $N(t) - \widetilde{N}(t)$ is a higher order infinitesimal of $t^{1/r}$, see [25,26]. SAs have been developed for various stochastic processes, such as random walks [27] and renewal-related processes [13,28]. There is a large volume of the literature using the SA to study queueing models, including the $GI/GI/1$ queue [18], $GI/GI/\infty$ queue [29], multiple channel queue [30], tandem-queue network [31], generalized Jackson network [19,32,33], non-preemptive priority queue [9], time-dependent Markovian network queues [34,35], and $(GI/GI)^K/1$/PPSD queue [7,18].

Organization of the rest of the paper We close this section by summarizing all notations used throughout the paper. In Sect. 2, we formally introduce the $(GI/GI)^K/1$/PPSD model and define the key performance functions. In Sect. 3, we review the FSLLN and fluid limit of the $(GI/GI)^K/1$/PPSD model because the fluid functions will be used to construct the prelimits of the LILs. In Sect. 4, we present our main results through Theorems 2–6. We also provide insights into these results. To substantiate the LIL results from an engineering perspective, we provide concrete numerical examples in Sect. 5. In Sect. 6, we give the proofs of the main results. Finally, in Sect. 7 we draw conclusions. Additional supporting materials, including extra numerical examples, omitted proofs, and alternative representations, appear in the Appendix.

Notations All random variables and processes are assumed to be defined on a common probability space $(\Omega, \mathcal{F}, \mathsf{P})$. We reserve $\mathsf{E}(\cdot)$ for expectation and $Var(\cdot)$ for variance. If two random variables X and Y have a common distribution then we write $X \stackrel{\text{d}}{=} Y$. The symbols $\mathbb{R}$ and $\mathbb{R}_+$ are used to denote the sets of real numbers and non-negative real numbers, respectively. For $a, b \in \mathbb{R}$, define $a \vee b \equiv \max\{a, b\}$ and $[a]^+ \equiv \max\{a, 0\}$. For a sequence $x_1, x_2, \ldots$, define $\sum_{i=k}^{j} x_i \equiv 0$ for $k > j$ (e.g., $\sum_{i=1}^{0} x_i = 0$). Let $\mathbb{C}$ be the space of continuous functions and $\mathbb{D}$ be the space of right-continuous functions with left limits. Define $\mathbb{D}_0 \equiv \{x \in \mathbb{D} : x(0) \geq 0\}$. Let $||f||_T \equiv \sup_{0 \leq t \leq T} |f(t)|$ be the uniform norm of f. We say $f_n \to f$ *uniformly on compact set* (u.o.c.) if $||f_n - f||_T \to \infty$, as $n \to \infty$. For two functions f and g, let $f \circ g(t) = f(g(t))$ denote the composition of f and g. We say $f(t) = O(g(t))$ as $t \to \infty$ if $\limsup_{t\to\infty} |f(t)/g(t)| \leq M$ for some $M > 0$ and $f(t) = o(g(t))$ as $t \to \infty$ if $\lim_{t\to\infty} |f(t)/g(t)| = 0$. We use the acronym "w.p.1." for "with probability one". Finally, we define the LIL scaling function

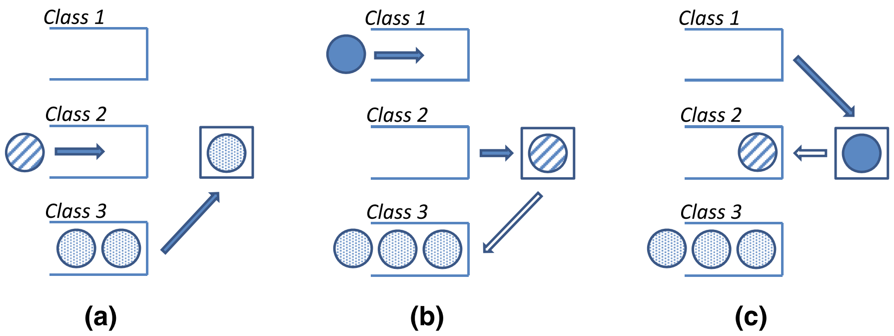

Fig. 1 The $(GI/GI)^3/1$/PPSD example **a** The server is serving a class-3 customer when there are no class-2 and class-3 customers in the system; **b** a newly arrived class-2 customer enters service preempting that class-3 customer who returns to the head of queue 3; and **c** a newly arrived class-1 customer enters service preempting that class-2 customer

$$\varphi(t) = \sqrt{2t \log \log t}.$$

2 The $(GI/GI)^K/1$/PPSD queueing model

The model consists of a single server and K queues, $K \geq 2$. Each queue k is fed by an external class-k arrival process, $1 \leq k \leq K$. In each queue, customers are served in the order of arrival. A preemptive priority service discipline (PPSD) is enforced among K classes: If a customer of higher priority arrives, the low-priority customer that is currently being served (if any) will be immediately bumped out of service and placed at the head of line of its own queue; after all customers of higher priorities leave the system, the server will resume serving that preempted customer until its service is completed or another interruption by a customer of higher priority. We label these classes from 1 to K with class 1 takes the highest priority, while class K the lowest. See Fig. 1 for an illustration.

For each class k, let $v_k(n)$ and $u_k(n)$ be the service time and interarrival time (time between two consecutive arrivals) of the nth customer. We assume that $u_k = \{u_k(n), n = 1, 2, \ldots\}$ and $v_k = \{v_k(n), n = 1, 2, \ldots\}$ are two independent i.i.d. sequence of non-negative random variables, having means $\mathsf{E}[u_k(1)] \equiv 1/\lambda_k$ and $\mathsf{E}[v_k(1)] \equiv 1/\mu_k$, variances $Var[v_k(1)]$ and $Var[v_k(1)]$, and *squared coefficients of variation* (SCV) $c^2_{a,k} \equiv Var[u_k(1)]/(\mathsf{E}[u_k(1)])^2$ and $c^2_{s,k} \equiv Var[v_k(1)]/(\mathsf{E}[v_k(1)])^2$, respectively. Define the partial sums

$$U_k(n) \equiv \sum_{i=1}^{n} u_k(i) \quad \text{and} \quad V_k(n) \equiv \sum_{i=1}^{n} v_k(i), \quad n = 1, 2, \ldots, \tag{4}$$

and their corresponding renewal processes

$$A_k(t) \equiv \max\{n \geq 0 : U_k(n) \leq t\} \text{ and } S_k(t) \equiv \max\{n \geq 0 : V_k(n) \leq t\}, \tag{5}$$

where $A_k(t)$ counts the total number of arrivals for class k customers in the time interval $[0, t]$ and $S_k(t)$ counts the number of class k customers the server can potentially serve in $[0, t]$ if there are no class i customers with $i < k$.

Define the overall traffic intensity

$$\rho \equiv \sum_{k=1}^{K} \rho_k \quad \text{where} \quad \rho_k \equiv \frac{\lambda_k}{\mu_k}, \quad k = 1, 2, \ldots, K. \tag{6}$$

We say the system is UL when $\rho < 1$, CL when $\rho = 1$, and OL when $\rho > 1$. Let $c_k^2 \equiv c_{a,k}^2 + c_{s,k}^2$ be the variability coefficient for class k (capturing the variabilities of both the arrival and service distributions). Let

$$\sigma_k^2 \equiv \sum_{j=1}^{k} \rho_j\, w_j, \quad \text{with} \quad w_j \equiv \frac{c_j^2}{\mu_j}.$$

Here σ_k^2 can be understood as the (weighted) cumulative utilization of service capacity by the first k classes.

Performance functions Let $Q_k(t)$ be the total number of class-k customers in the system at time t, let $Z_k(t)$ be the workload for class k at time t, that is the total amount of time required to process all class k customers assuming no future arrivals and no class $i < k$ customers after time t. Let $B_k(t)$ be the total amount of time the server is busy serving class k customers in $[0, t]$, that is $B_1(t) = \int_0^t \mathbf{1}_{\{Q_1(s)>0\}} \mathrm{d}s$ and

$$B_k(t) = \int_0^t \mathbf{1}_{\left\{Q_k(s)>0,\, Q_i(s)=0, i<k\right\}} \mathrm{d}s, \quad \text{for} \quad 2 \le k \le K. \tag{7}$$

Let $I_k(t)$ be the residual time in $[0, t]$ available to serve classes $k+1, \ldots, K$ after serving the first k classes, i.e.,

$$I_k(t) = t - \sum_{i=1}^{k} B_i(t). \tag{8}$$

Let $D_k(t) \equiv S_k(B_k(t))$ count the total number of class k customers that complete service by time t. We have

$$Q_k(t) = A_k(t) - S_k(B_k(t)) \ge 0, \tag{9}$$

$$Z_k(t) = V_k(A_k(t)) - B_k(t), \tag{10}$$

$$0 = \int_0^t Q_k(t) \mathrm{d}I_k(t), \tag{11}$$

where (9) holds by flow conservation, (10) holds because $V_k(A_k(t))$ represents the total amount of work (measured in time units) of class-k arrivals in $[0, t]$, and (11) implies that the idle process $I_k(t)$ increases only when $Q_k(t) = 0$.

The objective of the rest of the paper is to establish the LIL for performance functions $(Q_k, Z_k, B_k, I_k, D_k, 1 \le k \le K)$ and identify the LIL limits as functions of the model data

$$\mathcal{D} \equiv \Big(\lambda_k, \mu_k, c_{a,k}^2, c_{s,k}^2, c_k^2, \rho_k, \sigma_k, 1 \le k \le K\Big). \tag{12}$$

3 Fluid limits of the $(GI/GI)^K/1$/PPSD queue

Since the forms of the LILs involve the performance measures centered by their corresponding fluid functions, we next review the fluid limits of the $(GI/GI)^K/1$/PPSD model.

Let $Q \equiv (Q_1, \ldots, Q_K)$ be the vector of the queue length processes, also let Z, B, I, and D be the vectors of the workload, busy time, idle time, and departure processes in the same token. Define their LLN-scaled processes as

$$\bar{Q}^{(n)}(t) = \frac{1}{n}Q(nt), \quad \bar{Z}^{(n)}(t) = \frac{1}{n}Z(nt), \quad \bar{B}^{(n)}(t) = \frac{1}{n}B(nt),$$
$$\bar{I}^{(n)}(t) = \frac{1}{n}I(nt), \quad \bar{D}^{(n)}(t) = \frac{1}{n}D(nt).$$

We summarize the FSLLN and the fluid limits [8] in the next lemma, also see [6] for details and proofs.

Theorem 1 (FSLLN for the $(GI/GI)^K/1$/PPSD queue [8]) *Assume the system is initially empty. If* $\mathsf{E}[u_k(1)] < \infty$ *and* $\mathsf{E}[v_k(1)] < \infty$*, then*

$$\Big(\bar{Q}^{(n)}, \bar{Z}^{(n)}, \bar{B}^{(n)}, \bar{I}^{(n)}, \bar{D}^{(n)}\Big) \to \big(\bar{Q}, \bar{Z}, \bar{B}, \bar{I}, \bar{D}\big), \quad \textit{u.o.c., w.p.1}, \quad \textit{as } n \to \infty,$$

where $\bar{Q}, \bar{Z}, \bar{B}, \bar{I}$, *and* $\bar{D}$ *are* K*-dimensional deterministic vectors with their kth components satisfying*

$$\bar{Y}_k(t) \equiv \Psi(\bar{X}_k)(t), \quad \bar{Q}_k(t) \equiv \lambda_k t - \bar{D}_k(t) = \bar{X}_k(t) + \bar{Y}_k(t) = \Phi(\bar{X}_k)(t),$$
$$\bar{X}_k(t) \equiv (\lambda_k - \mu_k)t + \mu_k \sum_{l=1}^{k-1} \bar{B}_l(t), \quad \bar{B}_k(t) \equiv t - \sum_{l=1}^{k-1} \bar{B}_l(t) - \bar{I}_k(t),$$
$$\bar{I}_k(t) \equiv \frac{\bar{Y}_k(t)}{\mu_k}, \quad \bar{D}_k(t) \equiv \mu_k \bar{B}_k(t), \quad \bar{Z}_k(t) \equiv \frac{\bar{Q}_k(t)}{\mu_k}, \quad k = 1, \ldots, K, \tag{13}$$

and functions Φ *and* Ψ *are defined for* $x \in \mathbb{D}_0$ *as*

$$\Psi(x)(t) \equiv \sup_{0 \le s \le t} \{-x(s)\}^+ \quad \textit{and} \quad \Phi(x)(t) \equiv x(t) + \sup_{0 \le s \le t} \{-x(s)\}^+. \tag{14}$$

Remark 1 (Oblique reflection mapping) The mapping (Ψ, Φ) is known as the one-dimensional *oblique reflection mapping* (ORM), and is Lipschitz continuous in uniform norm; see [18] for detailed discussions. Alternative representations for (Ψ, Φ) are given in Appendix.

Remark 2 (Analytic solutions of the fluid functions) Equations in (13) uniquely define a set of deterministic and continuous functions $\left(\bar{Q}, \bar{Z}, \bar{B}, \bar{I}, \bar{D}\right)$ that are performance measures for the $(GI/GI)^K/1$/PPSD fluid model; see [8,36] for details. We provide the analytic solutions of the equations in (13) in Appendix.

4 Main results

In this section, we develop the LIL for the $(GI/GI)^K/1$/PPSD model in the following form:

$$\limsup_{T\to\infty} \frac{\left\|Q_k - \bar{Q}_k\right\|_T}{\varphi(T)} = Q_k^*, \quad \limsup_{T\to\infty} \frac{\left\|Z_k - \bar{Z}_k\right\|_T}{\varphi(T)} = Z_k^*,$$
$$\limsup_{T\to\infty} \frac{\left\|B_k - \bar{B}_k\right\|_T}{\varphi(T)} = B_k^*, \quad \limsup_{T\to\infty} \frac{\left\|I_k - \bar{I}_k\right\|_T}{\varphi(T)} = I_k^*,$$
$$\limsup_{T\to\infty} \frac{\left\|D_k - \bar{D}_k\right\|_T}{\varphi(T)} = D_k^*, \quad \text{w.p.1,} \quad \text{for} \quad k = 1, \ldots, K, \tag{15}$$

where $\left(\bar{Q}_k, \bar{Z}_k, \bar{B}_k, \bar{I}_k, \bar{D}_k\right)$ are the fluid functions defined in Sect. 3, and the LIL limits $\left(Q_k^*, Z_k^*, B_k^*, I_k^*, D_k^*\right)$ are explicit functions of the model data $\mathcal{D}$ in (12). Since the fluid functions have been widely used to approximate the mean values of their corresponding stochastic functions, these LIL limits refine the fluid approximation and provide an estimate (in the order of $\varphi(t)$) for the growth of the approximating error (deviation from the fluid paths).

We next present the LILs in three cases: (i) $\rho < 1$ (UL), (ii) $\rho = 1$ (CL), and (iii) $\rho > 1$ (OL) through Theorems 2–6. We give all proofs in Sect. 6. Throughout the rest of the paper, we assume that, for all $k = 1, \ldots, K$,

$$\mathsf{E}\left[u_k(1)^r\right] < \infty \quad \text{and} \quad \mathsf{E}\left[v_k(1)^r\right] < \infty \quad \text{for some } r > 2. \tag{16}$$

4.1 LILs for the UL and CL cases

We first establish the LIL for the UL case.

Theorem 2 (LIL for the UL $(GI/GI)^s K/1$/PPSD queue) *If $\rho < 1$, then the LIL (15) holds for $k = 1, 2, \ldots, K$ with*

$$Q_k^* = Z_k^* = 0, \quad B_k^* = \frac{c_k\sqrt{\lambda_k}}{\mu_k}, \quad I_k^* = \sigma_k, \quad \textit{and} \quad D_k^* = c_{a,k}\sqrt{\lambda_k}. \tag{17}$$

In addition, the limit superiors for Q and Z in (15) become limits.

Remark 3 (Understanding the LIL limits for the UL case) In the UL case, the whole queueing system is in light traffic. In an interval $[0, t]$ for large t, the server is idling asymptotically for a significant amount of time $(1-\rho)t > 0$. Hence the fluid queue length and workload vanish, i.e., $\bar{Q} = \bar{Z} = 0$ (see Appendix); also the queue length and workload of the queueing system are stochastically bounded, namely they do not grow with t at all. This explained why $Q^* = Z^* = 0$ in (17). In addition, the LILs for Q and Z hold in a strong sense with "$\limsup_{T\to\infty}$" upgraded to "$\lim_{T\to\infty}$" because $\|Q_k - \bar{Q}_k\|_T = \|Z_k - \bar{Z}_k\|_T = O(\log(T))$; see Sect. 6.

Interestingly, since the waiting times at all classes are asymptotically negligible, i.e., all customers are quickly served upon arrival, the class-k busy time B_k and service completion D_k become asymptotically independent with the performance of other classes. Hence B_k^* and D_k^* are determined only by parameters of class k: λ_k, μ_k, $c_{a,k}$, and $c_{s,k}$. In addition, D_k^*, capturing the asymptotic variability of the class-k departure, is solely characterized by λ_k and $c_{a,k}$ (parameters of the class-k arrival process), because the server is asymptotically always available when an arrival occurs, which makes the arrival process (rather than the service times) the deciding factor. The idle process I_k is determined by the busy times $B_1, \ldots, B_k$ so that I_k^* embodies parameters of all classes 1 to k through σ_k.

We next establish the LIL for the CL case.

Theorem 3 (LIL for the CL $(GI/GI)^K/1/\text{PPSD}$ queue) *If $\rho = 1$, then the LIL (15) holds with limits in (17) for $k = 1, 2, \ldots, K-1$ and*

$$Q_K^* = \mu_K \sigma_K, \quad D_K^* = \sqrt{\mu_K^2 \sigma_{K-1}^2 + \lambda_K c_{s,K}^2} \vee (\sqrt{\lambda_K} c_{a,K}^2),$$
$$Z_K^* = I_K^* = \sigma_K \quad \textit{and} \quad B_K^* = \frac{1}{\mu_K}\left[(\mu_K \sigma_{K-1}) \vee (\sqrt{\lambda_K} c_K)\right]. \tag{18}$$

Remark 4 (Understanding the LIL limits for the CL case) Due to the preemptive priority policy, the performance of class k is independent with that of all classes $j > k$. Since $\sum_{k=1}^{K-1} \rho_k = 1 - \rho_K < 1$, the subsystem consisting of the first $K-1$ classes can be treated as a UL $(GI/GI)^{K-1}/1/\text{PPSD}$ system. Hence, the LIL limits for the first $K-1$ classes coincide with those of the UL case in Theorem 2.

The class-K LIL limits are functions of the parameters of the first $K-1$ classes because class K takes the lowest priority and thus is influenced by the first $K-1$ classes. At the fluid level, the remaining service capacity for class K, $\rho_K = 1 - \sum_{k=1}^{K-1} \rho_k$, is barely enough to serve the class-K arrival. Note that the LIL limits for the departure process either are characterized only by $c_{a,K+1}$ and independent with all other parameters, if $c_{a,K+1}$ is large, or is independent with $c_{a,K+1}$ is if $c_{a,K+1}$ is small. Interestingly, only one of the two terms (i) the variability of class-K arrival and (ii) the sum of the variability of class-K service and those of the first $K-1$ classes, whichever is bigger, plays a role. We make a similar observation on the LIL limit of the busy period process.

4.2 LIL for the OL case

The OL case becomes more complicated. We next carefully treat three types of OL queues (in Theorems 4–6), categorized by the values of $\rho_1, \ldots, \rho_K$:

$$\textbf{Type}-\textbf{1}: \text{there exists a } k_0 : 1 \le k_0 < K \text{ such that } \sum_{j=1}^{k_0} \rho_j = 1 < \sum_{j=1}^{k_0+1} \rho_j; \tag{19}$$

$$\textbf{Type}-\textbf{2}: \text{there exists a } k_0 : 1 \le k_0 < K \text{ such that } \sum_{j=1}^{k_0} \rho_j < 1 < \sum_{j=1}^{k_0+1} \rho_j; \tag{20}$$

$$\textbf{Type}-\textbf{3}: \rho_1 > 1. \tag{21}$$

We point out that the two types in (19) and (20) have to be investigated separately because the LIL limits are not continuous in ρ at value 1.

Theorem 4 (LIL for the type-1 OL $(GI/GI)^K/1$/PPSD queue) *If the system is type-1 OL defined in (19), then the LIL (15) holds for all classes. First, the LIL limits for class k_0 satisfy (18) with K replaced by k_0. Second, the LIL limits for classes 1 to $k_0 - 1$ satisfy (17) with $k = 1, 2, \ldots, k_0 - 1$, if $k_0 > 1$. Third, for class $k_0 + 1$,*

$$Q^*_{k_0+1} = \sqrt{\lambda_{k_0+1} c^2_{a,k_0+1} + \mu^2_{k_0+1}\sigma^2_{k_0}}, \quad Z^*_{k_0+1} = \frac{\sqrt{\lambda_{k_0+1} c^2_{k_0+1} + \mu^2_{k_0+1}\sigma^2_{k_0}}}{\mu_{k_0+1}},$$
$$I^*_{k_0+1} = 0, \quad B^*_{k_0+1} = \sigma_{k_0}, \quad \textit{and} \quad D^*_{k_0+1} = \mu_{k_0+1}\sigma_{k_0}. \tag{22}$$

Last, if $k_0 + 1 < K$, then for $k = k_0 + 2, k_0 + 3, \ldots, K$,

$$Q^*_k = \sqrt{\lambda_k} c_{a,k}, \quad Z^*_k = \frac{c_k\sqrt{\lambda_k}}{\mu_k}, \quad \textit{and} \quad I^*_k = B^*_k = D^*_k = 0. \tag{23}$$

Remark 5 (Understanding the LIL limits for the type-1 OL case) Since the subsystem consisting the first k_0 queues can be treated as a CL $(GI/GI)^{k_0}/1$/PPSD queue, we refer to Remark 4 for interpretations of the LIL limits of the first k_0 classes. We next provide insights for classes $k_0 + 1, \ldots, K$.

We first explain the LIL limit of the queue length. The first term of $Q^*_{k_0+1}$ coincides with Q^*_k, $k \ge k_0 + 2$, representing the variability of the arrival process. Interestingly, Q^*_k is independent of the service SCV $c^2_{s,k}$ because customers from classes $k \ge k_0 + 1$ asymptotically will "never" enter service because the service capacity is completely occupied by the first k_0 classes. As a result, the queue length of class $k \ge k_0 + 1$ will grow to infinity. The second term of $Q^*_{k_0+1}$ is $\mu_{k_0+1}\sigma_{k_0} = \mu_{k_0+1} I^*_{k_0}$, which represents the influence from the first k_0 classes (recall that I_k is the residual time the server spends on serving low-priority classes after serving the first k_0 classes). Although the first k_0 classes have utilized all service capacity, it is still possible to serve some (perhaps very little) class k_0+1 customers. However, Q^*_k for $k \ge k_0+2$ does not have such a term and is thus independent of the first k_0 classes because customers there are indeed "never" served.

The comparison between $Z^*_{k_0+1}$ and Z^*_k ($k \geq k_0+2$) is similar. However, unlike the LIL limits for the queue length, Z^*_k depends on the service SCV because it involves $c_k = \sqrt{c^2_{a,k} + c^2_{s,k}}$, for $k \geq k_0+1$. This is so because the workload process keeps track of the total amount of unfinished service times, while the queue length process only counts the number of unfinished customers. Although these customers will "never" be served, their service variability will still make an impact to the workload: if service times are highly variable, it does not affect the queue lengths because no one will enter service, but it will make the workload process highly variable because a customer's service time will be added to the workload process immediately upon its arrival. An interesting observation is that Little's law (which holds for the fluid model, see Sect. 1) does not hold here (i.e., Z^*_k is not proportional to Q^*_k), unlike the UL and CL case.

To explain $D^*_{k_0+1} = \mu_{k_0+1} B^*_{k_0+1}$ and $B^*_{k_0+1} = B^*_{k_0} = \sigma_{k_0}$, we use a similar argument in Remark 4 because the busy times B_{k_0} and B_{k_0+1} again play a seesaw. Finally, since customers of class $k \geq k_0+2$ are indeed "never" served thus having 0 stochastic fluctuations, we have $I^*_{k_0+1} = I^*_k = B^*_k = D^*_k = 0$.

Theorem 5 (LIL for the type-2 OL $(GI/GI)^K/1/$PPSD queue) *If the system is type-2 OL defined in (20), then the LIL (15) holds for all classes. First, the LIL limits for classes 1 to k_0 satisfy (17) with $k = 1, 2, \ldots, k_0$. Second, for class k_0+1,*

$$
\begin{aligned}
Q^*_{k_0+1} &= \sqrt{\mu^2_{k_0+1}\sigma^2_{k_0} + \lambda_{k_0+1}c^2_{a,k_0+1} + \mu_{k_0+1}c^2_{s,k_0+1}\sqrt{1-\sum_{i=1}^{k_0}\rho_i}},\\
Z^*_{k_0+1} &= \sigma_{k_0+1}, \quad B^*_{k_0+1} = \sigma_{k_0}, \quad I^*_{k_0+1} = 0,\\
\textit{and} \quad D^*_{k_0+1} &= \sqrt{\mu^2_{k_0+1}\sigma^2_{k_0} + \mu_{k_0+1}c^2_{s,k_0+1}\sqrt{1-\sum_{l=1}^{k_0}\rho_l}}.
\end{aligned}
\tag{24}
$$

Last, if $k_0+1 < K$, then (23) holds for class $k = k_0+2, k_0+3, \ldots, K$.

Remark 6 (Understanding the LIL limits for the type-2 OL case) First, since the subsystem consisting the first k_0 queues can be treated as a UL $(GI/GI)^{k_0}/1/$PPSD queue, we refer to Remark 3 for explanations of the LIL limits of the first k_0 classes. Second, we refer to Remark 5 for explanations on LIL limits of classes k_0+2 to K because in both type-1 and type-2 OL cases, customers of these classes are indeed "never" served. Hence, the only complication (difference) appears in class k_0+1, which we exploit next.

The LIL limit of the queue length $Q^*_{k_0+1}$ is the square root of the sum of three terms, with the first term capturing the influence from the first k_0 classes through σ_{k_0}, the second term representing the variability of the arrival process, and the last term characterizing the variability of the departure process $D_{k_0+1}(t) = S_{k_0+1}(B_{k_0+1}(t))$ because as $t \to \infty$, $B_{k_0+1}(t)/t \to 1 - \sum_{i=1}^{k_0}\rho_i > 0$ is the long-run proportion of service capacity allocated to class $k_0 + 1$. Also note that $Q^*_{k_0+1}$ is not continuous in the traffic intensity at 1 (comparing to the type-1 OL case), namely, $Q^*_{k_0+1}$ does not coincide with that in (22) if we let $\sum_{i=1}^{k_0} \rho_i = 1$, which justifies the separation of type-1 and type-2 in the OL case. See Sect. 5 for more discussions on the discontinuity the LIL limits in the traffic intensity.

The LIL limit of the departure process $D^*_{k_0+1}$ can be understood similarly, except for the absence of the term $\lambda_{k_0+1} c^2_{a,k_0+1}$. This is so because the available service capacity is enough to serve only a proportion of class $k_0 + 1$ customers so that the queue length will go to infinity and neither the rate nor the variability of the arrival process makes an impact to $D^*_{k_0+1}$.

Little's law again fails to hold for the LIL limit of the workload $Z^*_{k_0+1}$, which captures both the variability of class $k_0 + 1$ and the influence from the first k_0 classes. The LIL limit of the busy time $B^*_{k_0+1} = \sigma_{k_0} = I^*_{k_0}$ because, $I_k(t)$, the remaining service capacity available for low-priority classes $k > k_0$, will asymptotically all be devoted to class $k_0 + 1$. Even so, it is still not enough to serve all class-$(k_0 + 1)$ customers, so it leaves no capacity at all for any class $k \geq k_0 + 2$ classes, which explains why $I^*_{k_0+1} = 0$.

Theorem 6 (LIL for the type-3 OL $(GI/GI)^K/1/$PPSD queue) *If the system is type-3 OL defined in (21), then the LIL (15) holds with limits in (23) for* $k = 2, 3, \ldots, K$ *and*

$$Q^*_1 = \sqrt{\lambda_1 c^2_{a,1} + \mu_1 c^2_{s,1}}, \ Z^*_1 = \sigma_1, \ B^*_1 = I^*_1 = 0, \ \text{and} \ D^*_1 = \mu^{1/2}_{s,1} c_{s,1}. \tag{25}$$

Remark 7 (Understanding the LIL limits for the type-3 OL case) Theorem 6 supplements Theorem 5. We can simply treat class 1 as the class k_0+1 of the type-2 OL case (namely, setting $k_0 = 0$ so that $\sigma_{k_0} = 0$ in Theorem 5). Since class 1 will be partly served while classes 2 to K will "never" get served, we refer to Remark 6 for detailed discussions. Although Theorem 6 can be viewed as a special case of Theorem 5, we present the type-3 OL case separately to avoid complications in notations.

5 Numerical examples

Before presenting the proofs, we next consider two numerical examples to obtain insights of the LIL limits in Theorem 2–6.

5.1 Discontinuities in the queue index k

Due to the PPSD service discipline, the LIL limits of queues establish discontinuities around k_0. We next demonstrate this jump structure.

Example 1 (Discontinuity of the LIL limits in the class index k) We first consider a $(GI/GI)^6/1/$PPSD model with $K = 6$, $\lambda_k = 0.3$, $\mu_k = c_{a,k} = c_{s,k} = 1$ for all $1 \leq k \leq K = 6$. This example belongs to the type-2 OL case because $k_0 = 3$, $\sum_{k=1}^{3} \rho_k = 0.9 < 1$, and $\sum_{k=1}^{4} \rho_k = 1.2 > 1$.

Table 1 The LIL limits for Example 1

k	1	2	3	4	5	6
Q_k^*	0	0	0	$\sqrt{2.1+\sqrt{0.1}}$	$\sqrt{0.3}$	$\sqrt{0.3}$
Z_k^*	0	0	0	$\sqrt{2.4}$	$\sqrt{0.6}$	$\sqrt{0.6}$
B_k^*	$\sqrt{0.6}$	$\sqrt{0.6}$	$\sqrt{0.6}$	$\sqrt{1.8}$	0	0
I_k^*	$\sqrt{0.6}$	$\sqrt{1.2}$	$\sqrt{1.8}$	0	0	0
D_k^*	$\sqrt{0.3}$	$\sqrt{0.3}$	$\sqrt{0.3}$	$\sqrt{1.8+\sqrt{0.1}}$	0	0

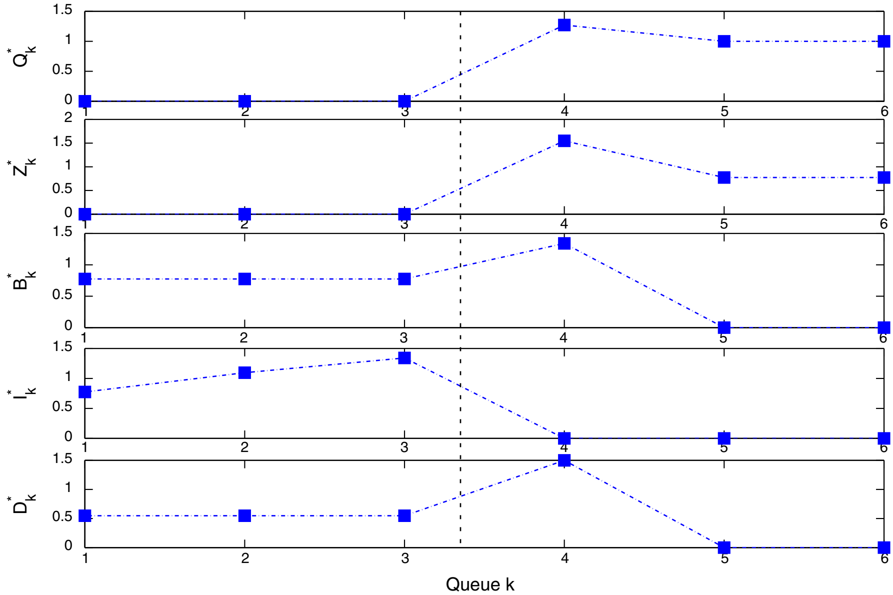

Fig. 2 LIL limits of Example 1 as functions of k, $1 \le k \le 6$, with $k_0 = 3$, $\sum_{k=1}^{3} \rho_k < 1$ and $\sum_{k=1}^{4} \rho_k > 1$

According to Theorem 5, we compute the LIL limits in Table 1 and plot these limits as functions of k in Fig. 2. The vertical line in 2 serving as a "benchmark" for $\rho = 1$ separates queue 3 and queue 4 (because $\sum_{k=1}^{3} \rho_k = 0.9 < 1$ and $\sum_{k=1}^{4} \rho_k = 1.2 > 1$). We see that all LIL limits jump at k_0 and $k_0 + 1$. Similarly, the LIL limits Q_k^*, Z_k^*, B_k^* and D_k^* all peak at $k = k_0 + 1 = 4$, with stochastic processes Q_k, Z_k, B_k and D_k experiencing the largest asymptotical variability at $k = k_0 + 1$. The LIL limit I_k^* increases in k, peaks at $k_0 = 3$ and then drops to 0, because the variability of I_k is cumulative (thus increasing) for $1 \le k \le k_0$ and then becomes asymptotically negligible for all $k_0 < k \le K$. See Remark 6 for more discussions.

We consider a second example in the Appendix to explore the discontinuity in k for the type-1 OL model. See Table 2 and Fig. 4 in Appendix.

Table 2 The LIL limits for Example 3

k	1	2	3	4	5	6
Q_k^*	0	0	$3\sqrt{2/3}$	$\sqrt{7}$	1	1
Z_k^*	0	0	$\sqrt{2/3}$	$\sqrt{8/3}$	$\sqrt{2}/3$	$\sqrt{2}/3$
B_k^*	$\sqrt{2}/3$	$\sqrt{2}/3$	$2/3$	$\sqrt{2/3}$	0	0
I_k^*	$\sqrt{2}/3$	$2/3$	$\sqrt{2/3}$	0	0	0
D_k^*	1	1	$\sqrt{5}$	$3\sqrt{2/3}$	0	0

5.2 Sensitivity to the traffic intensity ρ

We have exhibited different structures for the LIL limits in Theorems 2–6 that cover all cases (UL, CL, and OL) categorized by different values of the traffic intensity ρ. We find it impossible to unify these cases into one framework because our formulas are in different form and our results provide distinct implications. On the one hand, this disparity can be understood through the distinct proof technique for each category (see Sect. 6); on the other hand, we conduct another numerical example to understand the discontinuity of these LIL limits in the traffic intensity ρ.

Example 2 (Discontinuity of the LIL limits in the traffic intensity) Let $K = 4$, $\lambda_1 = \lambda_3 = \lambda_4 = 0.5$, $\mu_k = c_{a,k} = c_{s,k} = 1$ for all $k = 1, 2, 3, 4$. But we will vary $\rho_2 = \lambda_2/\mu_2 = \lambda_2$ in an interval $[0, 1]$. The idea is to increase ρ_2 so that we can walk through all cases considered in Theorems 2–6.

We have four cases: (i) $\rho_1 < 1$, $\rho_1 + \rho_2 < 1$ and $\rho_1 + \rho_2 + \rho_3 = 1$ when $\rho_2 = 0$ (with $k_0 = 3$ for the case in Theorem 4); (ii) $\rho_1 < 1$, $\rho_1+\rho_2 < 1$ and $\rho_1+\rho_2+\rho_3 > 1$ when $\rho_2 \in (0, 0.5)$ (with $k_0 = 2$ for the case in Theorem 5); (iii) $\rho_1 < 1$, $\rho_1 + \rho_2 = 1$ and $\rho_1 + \rho_2 + \rho_3 > 1$ when $\rho_2 = 0.5$ (with $k_0 = 2$ for the case in Theorem 4); and (iv) $\rho_1 < 1$, $\rho_1 + \rho_2 > 1$ and $\rho_1 + \rho_2 + \rho_3 > 1$ when $\rho_2 \in (0.5, 1]$ (with $k_0 = 1$ for the case in Theorem 5).

Since the LIL limits for classes 1 and 4 are relatively simple, we plot the LIL limits of classes 2 and 3 as functions of ρ_2 in Fig. 3. In Sect. 1, we provide complete formulas for all LIL limits as functions of ρ_2. We see that the limits are neither right continuous nor left continuous. The variabilities are non-decreasing functions in ρ_2 for $0 < \rho_2 < 1/2$, representing the type-2 OL case with k_0. When $\rho_2 = 1/2$, jumps occur because the system now switches to the type-2 OL case with $k_0 = 2$. These functions jump again after value $1/2$ and are non-decreasing in the interval $(1/2, 1]$. This example clearly explains why all these cases have to be treated separately as in Theorems 2–6.

6 Proofs of main results

In this section, we prove the main results, namely the LIL (15) in UL, CL, and OL cases given in Theorems 2–6. We first present some preliminary results that are building

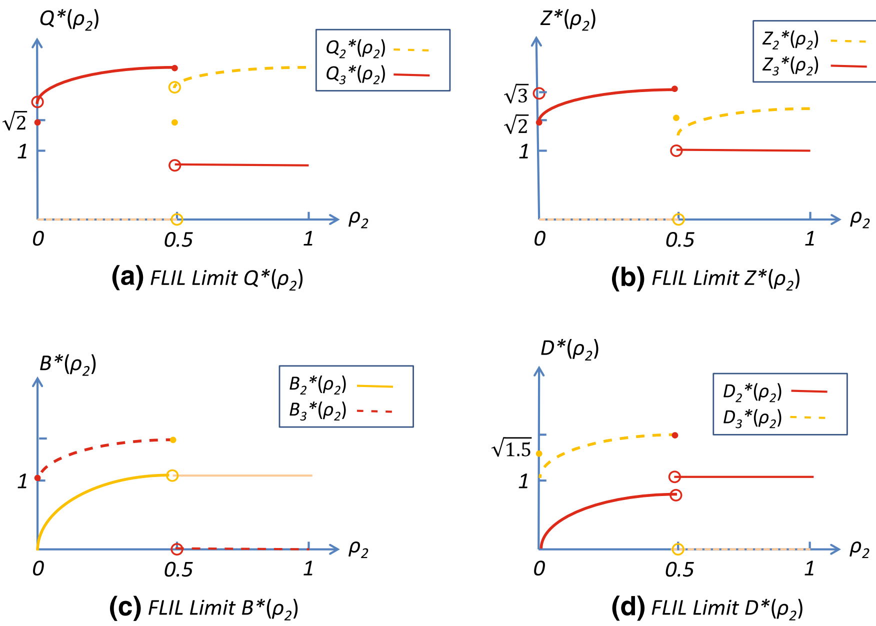

Fig. 3 The LIL limits Q_k^*, Z_k^*, B_k^* and D_k^*, $k = 2, 3$, as functions of ρ_2. We omit I_k^* which is relatively simple

blocks for the proofs. We introduce results related to SAs of the performance functions for the $(GI/GI)^K/1$/PPSD system in Sect. 6.1. We next prove the LILs in UL, CL, and OL cases in Sects. 6.2–6.6, applying these preliminary results.

6.1 Strong approximations and related results

The idea of the SA is to approximate a discrete process, such as the queue length Q, by the sum of two continuous functions: (i) the deterministic fluid function $\bar{Q}$ and (ii) standard BMs, with $\bar{Q}$ characterizing the mean value and the BMs quantifying the stochastic fluctuations around that mean value. We next introduce the SAs for the $(GI/GI)^K/1$/PPSD system.

Lemma 1 (Strong approximations for $(GI/GI)^K/1/$ PPSD) *If (16) holds, then for $k = 1, 2, \ldots, K$,*

$$\left\| Q_k - \widetilde{Q}_k \right\|_T = o(T^{1/r}), \quad \left\| Z_k - \widetilde{Z}_k \right\|_T = o(T^{1/r}), \quad \left\| B_k - \widetilde{B}_k \right\|_T = o(T^{1/r}),$$
$$\left\| I_k - \widetilde{I}_k \right\|_T = o(T^{1/r}), \quad \left\| D_k - \widetilde{D}_k \right\|_T = o(T^{1/r}), \quad \textit{w.p.1.} \tag{26}$$

where

$$\widetilde{Q}_k(t) \equiv \widetilde{X}_k(t) + \widetilde{Y}_k(t) = \Phi(\widetilde{X}_k)(t), \quad \widetilde{Y}_k(t) \equiv \Psi(\widetilde{X}_k)(t),$$

$$\widetilde{X}_k(t) \equiv (\lambda_k - \mu_k)t + \mu_k \sum_{l=1}^{k-1} \widetilde{B}_l(t) + \widetilde{W}_k(t),$$

$$\widetilde{B}_k(t) \equiv t - \sum_{j=1}^{k-1} \widetilde{B}_j(t) - \widetilde{I}_k(t), \quad \widetilde{I}_k(t) \equiv \frac{1}{\mu_k} \widetilde{Y}_k(t),$$

$$\widetilde{Z}_k(t) \equiv \frac{1}{\mu_k} \widetilde{Q}_k(t) + \frac{1}{\mu_k} \left[\mu_k^{1/2} c_{s,k} W_{s,k}(\bar{B}_k(t)) - \mu_k^{1/2} c_{s,k} W_{s,k}(\rho_k t) \right],$$

$$\widetilde{D}_k(t) \equiv \mu_k \widetilde{B}_k(t) + \mu_k^{1/2} c_{s,k} W_{s,k}(\bar{B}_k(t)),$$

$$\widetilde{W}_k(t) \equiv \lambda_k^{1/2} c_{a,k} W_{a,k}(t) - \mu_k^{1/2} c_{s,k} W_{s,k}(\bar{B}_k(t)), \tag{27}$$

$W_{a,k}$ and $W_{s,k}$ are independent standard BMs associated with the arrival and service processes of class k, respectively, and Ψ and Φ are defined in (14).

Proof Complementing Chen and Shen [7] which establishes the SAs for Q_k, Z_k, and I_k, we provide the SAs for $\widetilde{B}_k$ and $\widetilde{D}_k$.

We first prove the SA for B_k. The SA for I_k, namely $\|I_k - \widetilde{I}_k\|_T = o(T^{1/r})$, together with (8) and the third equation in (27), implies that

$$\|B_k - \widetilde{B}_k\|_T = o(T^{1/r}), \quad \text{w.p.1.} \tag{28}$$

Since the system is assumed to be empty initially, we note that $\widetilde{X}_k(0) = 0$ for all $k = 1, 2, \ldots, K$. The sixth equation of (27) implies that

$$\begin{aligned} D_k(t) - \widetilde{D}_k(t) &= S_k(B_k(t)) - \widetilde{D}_k(t) \\ &= \left[S_k(B_k(t)) - \mu_k B_k(t) - \mu_k^{1/2} c_{s,k} W_{s,k}(B_k(t)) \right] \\ &\quad + \mu_k (B_k(t) - \widetilde{B}_k(t)) \\ &\quad + \mu_k^{1/2} c_{s,k} \left[W_{s,k}(B_k(t)) - W_{s,k}(\bar{B}_k(t)) \right]. \end{aligned}$$

Following Theorem 3.4 in [7], we have $\|B_k - \bar{B}_k\|_T = O(\varphi(T))$ w.p.1., which together with Lemma 6.21 in [18] implies that

$$\|W_{s,k}(B_k) - W_{s,k}(\bar{B}_k)\|_T = o(T^{1/r}), \quad \text{w.p.1.}$$

Therefore, (28) and SA of renewal processes (see Theorem 5.14 in [18]) conclude that $\|D_k - \bar{D}_k\|_T = o(T^{1/r})$. □

We next provide two corollaries following Lemma 1.

Corollary 1 *For $k = 1, 2, \ldots, K$,*

$$\widetilde{X}_k(t) - \bar{X}_k(t) = -\sum_{l=1}^{k-1} \frac{\mu_k}{\mu_l} \left[\widetilde{Q}_l(t) - \bar{Q}_l(t) \right] + W_k(t), \tag{29}$$

$$\bar{B}_k(t) - \widetilde{B}_k(t) = \frac{1}{\mu_k}\left[\widetilde{Q}_k(t) - \bar{Q}_k(t)\right] - \frac{1}{\mu_k}\widetilde{W}_k(t), \tag{30}$$

where $W_k(t) \equiv \mu_k \sum_{l=1}^{k} \widetilde{W}_l(t)/\mu_l$, $\widetilde{W}_l$ *is defined in* (27), $k = 1, 2, \ldots, K$.

Using the SAs, Corollary 1 expresses $\widetilde{X}_k$ as a function of $\widetilde{Q}_1, \widetilde{Q}_2, \ldots, \widetilde{Q}_{k-1}$. Because $\widetilde{Q}_k = \Phi(\widetilde{X}_k)$ and $\widetilde{I}_k = \Psi(\widetilde{X}_k)/\mu_k$ (see (27)), we exploit an inductive argument to find Q_k^* using $Q_1^*, Q_2^*, \ldots, Q_{k-1}^*$. We obtain B_k^* (as a function of Q_k^*) through (30). Corollary 1 will be used in the proofs of Theorems 2–5.

Proof We prove Corollary 1 by induction. We first show that (29) and (30) hold for class $k = 1$. By (27), we have

$$\widetilde{X}_1(t) = \bar{X}_1(t) + W_1(t) \quad \text{and}$$
$$\bar{B}_1(t) - \widetilde{B}_1(t) = \left(\bar{B}_1(t) - t\right) + \widetilde{I}_1(t) = \left(\bar{B}_1(t) - t\right) + \frac{1}{\mu_1}\left[\widetilde{Q}_1(t) - \widetilde{X}_1(t)\right].$$

If $\rho_1 \le 1$, then $\bar{B}_1(t) = \rho_1 t$, $\bar{X}_1(t) = (\lambda_1 - \mu_1)t \le 0$, $\bar{Q}_1(t) = 0$, and $\bar{B}_1(t) - t = \bar{X}_1(t)/\mu_1$ (see Theorem 1 and Sect. 1). Hence

$$\begin{aligned}\bar{B}_1(t) - \widetilde{B}_1(t) &= \frac{1}{\mu_1}\bar{X}_1(t) + \frac{1}{\mu_1}\left\{\widetilde{Q}_1(t) - \left[\bar{X}_1(t) + W_1(t)\right]\right\}\\ &= \frac{1}{\mu_1}\left[\widetilde{Q}_1(t) - \bar{Q}_1(t)\right] - \frac{1}{\mu_1}\widetilde{W}_1(t).\end{aligned}$$

If $\rho_1 > 1$, then $\bar{B}_1(t) = t$ and $\bar{Q}_1(t) = (\lambda_1 - \mu_1)t = \bar{X}_1(t)$, so that

$$\begin{aligned}\bar{B}_1(t) - \widetilde{B}_1(t) &= \frac{1}{\mu_1}\left[\widetilde{Q}_1(t) - \widetilde{X}_1(t)\right]\\ &= \frac{\widetilde{Q}_1(t) - \left[\bar{X}_1(t) + \widetilde{W}_1(t)\right]}{\mu_1} = \frac{\widetilde{Q}_1(t) - \bar{Q}_1(t)}{\mu_1} - \frac{\widetilde{W}_1(t)}{\mu_1}.\end{aligned}$$

We next assume that (29) and (30) hold for $1, 2, \ldots, k$. For class $k+1$,

$$\begin{aligned}\widetilde{X}_{k+1}(t) &= (\lambda_{k+1} - \mu_{k+1})t + \mu_{k+1}\sum_{l=1}^{k}\widetilde{B}_l(t) + \widetilde{W}_{k+1}(t)\\ &= (\lambda_{k+1} - \mu_{k+1})t + \mu_{k+1}\sum_{l=1}^{k}\bar{B}_l(t) - \mu_{k+1}\sum_{l=1}^{k}\left[\bar{B}_l(t) - \widetilde{B}_l(t)\right] + \widetilde{W}_{k+1}(t)\\ &= \bar{X}_{k+1}(t) + \widetilde{W}_{k+1}(t) - \mu_{k+1}\sum_{l=1}^{k}\left\{\frac{1}{\mu_l}\left[\widetilde{Q}_l(t) - \bar{Q}_l(t)\right] - \frac{1}{\mu_l}\widetilde{W}_l(t)\right\}\\ &= \bar{X}_{k+1}(t) - \sum_{l=1}^{k}\frac{\mu_{k+1}}{\mu_l}\left[\widetilde{Q}_l(t) - \bar{Q}_l(t)\right] + W_{k+1}(t),\end{aligned} \tag{31}$$

where the first equality follows from (27) and the third equality holds by (13) and the induction hypothesis.

$$
\begin{aligned}
&\bar{B}_{k+1}(t)-\widetilde{B}_{k+1}(t)\\
&=\bar{B}_{k+1}(t)-\left[t-\sum_{l=1}^{k}\widetilde{B}_l(t)-\widetilde{I}_{k+1}(t)\right]\\
&=\left(\bar{B}_{k+1}(t)-t\right)-\sum_{l=1}^{k}\left[\bar{B}_l(t)-\widetilde{B}_l(t)\right]+\sum_{l=1}^{k}\bar{B}_l(t)+\widetilde{I}_{k+1}(t)\\
&=\left[\sum_{l=1}^{k+1}\bar{B}_l(t)-t\right]-\sum_{l=1}^{k}\left\{\frac{\widetilde{Q}_l(t)-\bar{Q}_l(t)}{\mu_l}-\frac{\widetilde{W}_l(t)}{\mu_l}\right\}+\frac{\widetilde{Q}_{k+1}-\widetilde{X}_{k+1}(t)}{\mu_{k+1}}\\
&=\left[\sum_{l=1}^{k+1}\bar{B}_l(t)-t\right]-\frac{\bar{X}_{k+1}(t)}{\mu_{k+1}}+\frac{\widetilde{Q}_{k+1}(t)}{\mu_{k+1}}-\frac{\widetilde{W}_{k+1}(t)}{\mu_{k+1}}\\
&=-\frac{1}{\mu_{k+1}}\left[\lambda_{k+1}t-\bar{D}_{k+1}(t)\right]+\frac{\widetilde{Q}_{k+1}(t)}{\mu_{k+1}}-\frac{\widetilde{W}_{k+1}(t)}{\mu_{k+1}}\\
&=\frac{1}{\mu_{k+1}}\left[\widetilde{Q}_{k+1}(t)-\bar{Q}_{k+1}(t)\right]-\frac{1}{\mu_{k+1}}\widetilde{W}_{k+1}(t),
\end{aligned}
$$

where the first equality follows from (27), the third equality follows from the induction hypothesis and (27), the fourth equality holds by (31), and the last two equalities hold by (13). □

Because $T^{1/r}=o(\varphi(T))$ for some $r>2$, following Lemma 1 we have the next Corollary, which transforms the original LIL of the performance functions (defined in (15)) to the equivalent LIL of their corresponding SAs. Its proof is omitted.

Corollary 2 *If (16) holds, then for* $k=1,2,\ldots,K$,

$$
\begin{aligned}
\limsup_{T\to\infty}\frac{\|Q_k-\bar{Q}_k\|_T}{\varphi(T)}&=\limsup_{T\to\infty}\frac{\|\widetilde{Q}_k-\bar{Q}_k\|_T}{\varphi(T)},\\
\limsup_{T\to\infty}\frac{\|Z_k-\bar{Z}_k\|_T}{\varphi(T)}&=\limsup_{T\to\infty}\frac{\|\widetilde{Z}_k-\bar{Z}_k\|_T}{\varphi(T)},\\
\limsup_{T\to\infty}\frac{\|B_k-\bar{B}_k\|_T}{\varphi(T)}&=\limsup_{T\to\infty}\frac{\|\widetilde{B}_k-\bar{B}_k\|_T}{\varphi(T)},\\
\limsup_{T\to\infty}\frac{\|I_k-\bar{I}_k\|_T}{\varphi(T)}&=\limsup_{T\to\infty}\frac{\|\widetilde{I}_k-\bar{I}_k\|_T}{\varphi(T)},\\
\limsup_{T\to\infty}\frac{\|D_k-\bar{D}_k\|_T}{\varphi(T)}&=\limsup_{T\to\infty}\frac{\|\widetilde{D}_k-\bar{D}_k\|_T}{\varphi(T)},\quad \textit{w.p.1}.
\end{aligned}
\tag{32}
$$

Given Corollary 2, it remains to prove that the right-hand sides of (32) agree with the LIL limits in (15).

6.2 Proof of Theorem 2

The next two results will be used in the proof of Theorem 2. Lemma 2 provides an exponential bound for the tail probability of the extreme value of the queue length, namely $\sup_{0\le t\le T}\widetilde{Q}_k(t)$. Because the proof of Lemma 2 is similar to the proof of Theorem 3.1 in [37], we omit it here but give it in the Appendix. Using this exponential bound, we can easily establish Corollary 3, which implies that the extreme value of the queue length is $O(\log T)$ as $T\to\infty$.

Lemma 2 (Exponential bounds for the UL queue lengths) *If* $\rho<1$, *then for any* $z\ge 0$ *and* $k=1,\dots,K$,

$$\mathsf{P}\left\{\sup_{0\le t\le T}\widetilde{Q}_k(t)\ge z\right\}\le N_k\,\exp\left\{-2\gamma\prod_{j=1}^{k}\delta_j\,z\right\},\tag{33}$$

where

$$\delta_1\equiv 1,\quad N_1\equiv 1,\quad N_k\equiv 2+\sum_{l=2}^{k-1}N_l,\quad \delta_k\equiv\frac{\min\{\mu_1,\dots,\mu_k\}}{2(k-1)\mu_k},\quad k=2,\dots,K,$$
$$\gamma\equiv\min\left\{\frac{\theta_1}{\mu_1^2\sigma_1^2},\dots,\frac{\theta_K}{\mu_K^2\sigma_K^2}\right\}\quad\text{and}\quad\theta_k\equiv\mu_k\left(1-\sum_{l=1}^{k}\rho_l\right),\quad k=1,\dots,K.$$

Corollary 3 *If* $\rho<1$, *then for all* $k=1,2,\dots,K$,

$$\left\|\widetilde{Q}_k\right\|_T=O(\log T),\quad w.p.1.\tag{34}$$

Proof Lemma 1 implies that $\widetilde{Q}_k(t)\ge 0$ for all $t\ge 0$ and $k=1,2,\dots,K$. Letting $z=\log T/(\gamma\prod_{j=1}^{k}\delta_j)$ in (33) yields that

$$\mathsf{P}\left(\sup_{0\le t\le T}\widetilde{Q}_k(t)\ge\frac{\log T}{\gamma\prod_{j=1}^{k}\delta_j}\right)\le N_k\frac{1}{T^2}.$$

Hence, Borel-Cantelli lemma implies that $\left\|\widetilde{Q}_k\right\|_T=O(\log T)$ w.p.1. for all $k=1,\dots,K$. □

Proof of Theorem 2 We now prove Theorem 2 using Lemma 2 and Corollary 3. We consider a UL system. First, for $k=1,2,\dots,K$, we observe that $\bar{Q}_k(t)=\bar{Z}_k(t)=0$ and $\bar{B}_k(t)=\rho_k t$ (see Sect. 1 for details), so that $\widetilde{Z}_k(t)=\widetilde{Q}_k(t)/\mu_k$ in (27). Hence, the LIL for Q_k holds as a simple consequence of (34) with $Q_k^*=0$ and the LIL for Z_k holds with $Z_k^*=0$.

We next prove the LILs for B_k and I_k. Since $\widetilde{W}_k(t)$ now becomes a driftless BM with variance $\lambda_k c_k^2$, it follows from (30) and (2) that

$$\limsup_{T\to\infty}\frac{\|\bar{B}_k-\widetilde{B}_k\|_T}{\varphi(T)}=\frac{1}{\mu_k}\limsup_{T\to\infty}\frac{\|\widetilde{W}_k\|_T}{\varphi(T)}=\frac{c_k\sqrt{\lambda_k}}{\mu_k},\quad \text{w.p.1.}$$

Regarding the LIL for I_k, (27) and (30) imply that

$$\begin{aligned}\bar{I}_k(t)-\widetilde{I}_k(t)&=\left(t-\sum_{l=1}^{k}\bar{B}_l(t)\right)-\left(t-\sum_{l=1}^{k}\widetilde{B}_l(t)\right)\\&=-\sum_{l=1}^{k}\left(\bar{B}_l(t)-\widetilde{B}_l(t)\right)=-\sum_{l=1}^{k}\frac{1}{\mu_l}\left[\widetilde{Q}_l(t)-\bar{Q}_l(t)\right]+\sum_{l=1}^{k}\frac{1}{\mu_l}\widetilde{W}_l(t).\end{aligned}$$

The LIL for I_k holds then with $I_k^*=\sigma_k$ because $Q_k^*=0$ and $\sum_{l=1}^{k}\widetilde{W}_l(t)/\mu_l$ is a driftless BM with variance parameter σ_k.

Finally, (30) implies that

$$\begin{aligned}\widetilde{D}_k(t)-\bar{D}_k(t)&=\mu_k^{1/2}c_{s,k}W_{s,k}(\bar{B}_k(t))+\mu_k(\widetilde{B}_k(t)-\bar{B}_k(t))\\&=\mu_k^{1/2}c_{s,k}W_{s,k}(\bar{B}_k(t))-\left[\widetilde{Q}_k(t)-\widetilde{W}_k(t)\right]\\&=-\widetilde{Q}_k(t)+\lambda_k^{1/2}c_{a,k}W_{a,k}(t).\end{aligned}$$

This, together with $Q_k^*=0$ and (2), completes the proof of the LIL for D_k with $D_k^*=c_{a,k}\sqrt{\lambda_k}$. □

6.3 Proof of Theorem 3

We next prove Theorem 3 for a CL $(GI/GI)^K/1$/PPSD system with $\rho=1$. First, it is easy to see that the LIL in (15) holds with limits in (17) for $k=1,2,\ldots,K-1$, because the first $K-1$ classes form a UL $(GI/GI)^{K-1}/1$/PPSD system (since $\sum_{l=1}^{K-1}\rho_l<1$). Hence, it suffices to prove the LIL for class K with limits in (18). We do so by exploiting the next three Lemmas.

Lemma 3 (Generalizing the LIL of one BM) *Suppose $\widehat{W}_1(t)$ and $\widehat{W}_2(t)$ are two independent standard BMs, $\bar{\sigma}_1>0$ and $\bar{\sigma}_2>0$ are two constants, then w.p.1*

$$\limsup_{T\to\infty}\frac{\sup_{0\le t\le T}\left|\bar{\sigma}_1\widehat{W}_1(t)+\sup_{0\le s\le t}\bar{\sigma}_2\widehat{W}_2(s)\right|}{\varphi(T)}=\sqrt{\bar{\sigma}_1^2+\bar{\sigma}_2^2}. \tag{35}$$

Proof Notice that

$$\sup_{0\le t\le T}\left|\bar{\sigma}_1\widehat{W}_1(t)+\sup_{0\le s\le t}\bar{\sigma}_2\widehat{W}_2(s)\right|=\sup_{0\le t\le T}\left|\bar{\sigma}_1\widehat{W}_1(t)+\sup_{0\le r\le 1}\bar{\sigma}_2\widehat{W}_2(rt)\right|$$

$$
\begin{aligned}
&= \sup_{0\le t\le T}\left|\sup_{0\le r\le 1}\left[\bar\sigma_1\widehat W_1(t)+\bar\sigma_2\widehat W_2(rt)\right]\right| \\
&\overset{d}{=} \sup_{0\le t\le T}\left|\sup_{0\le r\le 1}\left[\bar\sigma_1\widehat W_1(t)+\sqrt{r}\bar\sigma_2\widehat W_2(t)\right]\right| \\
&\overset{d}{=} \sup_{0\le t\le T}\left|\sup_{0\le r\le 1}(\sqrt{\bar\sigma_1^2+r\bar\sigma_2^2})\widehat W_1(t)\right|,
\end{aligned}
$$

where the second equality in distribution holds because $\widehat W_1$ and $\widehat W_2$ are independent standard BMs. So,

$$
\begin{aligned}
\limsup_{T\to\infty}\frac{\sup_{0\le t\le T}\left|\bar\sigma_1\widehat W_1(t)+\sup_{0\le s\le t}\bar\sigma_2\widehat W_2(s)\right|}{\varphi(T)} &= \limsup_{T\to\infty}\frac{\sup_{0\le t\le T}\left|\sup_{0\le r\le 1}(\sqrt{\bar\sigma_1^2+r\bar\sigma_2^2})\widehat W_1(t)\right|}{\varphi(T)} \\
&= \sup_{0\le r\le 1}(\sqrt{\bar\sigma_1^2+r\bar\sigma_2^2}) = \sqrt{\bar\sigma_1^2+\bar\sigma_2^2}.
\end{aligned}
$$

□

Lemma 4 (Generalizing the LIL of one BM) *Suppose that $\widehat W_1(t)$ and $\widehat W_2(t)$ are two independent standard BMs, $\bar\sigma_1>0, \bar\sigma_2>0$ are two constants, then w.p.1*

$$
\limsup_{T\to\infty}\frac{\sup_{0\le t\le T}\left|\sup_{0\le s\le t}\left[\bar\sigma_1\widehat W_1(t-s)-\bar\sigma_2\widehat W_2(s)\right]\right|}{\varphi(T)} = \bar\sigma_1\vee\bar\sigma_2. \tag{36}
$$

Proof First,

$$
\begin{aligned}
&\sup_{0\le t\le T}\left|\sup_{0\le s\le t}\left[\bar\sigma_1\widehat W_1(t-s)-\bar\sigma_2\widehat W_2(s)\right]\right| \\
&= \sup_{0\le t\le T}\left|\sup_{r\in[0,1]}\left[\bar\sigma_1\widehat W_1((1-r)t)-\bar\sigma_2\widehat W_2(rt)\right]\right| \\
&\overset{d}{=} \sup_{0\le t\le T}\left|\sup_{r\in[0,1]}\left[\sqrt{1-r}\bar\sigma_1\widehat W_1(t)-\sqrt{r}\bar\sigma_2\widehat W_2(t)\right]\right| \\
&\overset{d}{=} \sup_{0\le t\le T}\left|\sup_{r\in[0,1]}\sqrt{(1-r)\bar\sigma_1^2+r\bar\sigma_2^2}\,\widehat W(t)\right|,
\end{aligned}
$$

where $\widehat{W}$ is a standard BM. Next, we have

$$\limsup_{T\to\infty} \frac{\sup\limits_{0\le t\le T}\left|\sup\limits_{0\le s\le t}\left[\bar{\sigma}_1\widehat{W}_1(t-s)-\bar{\sigma}_2\widehat{W}_2(s)\right]\right|}{\varphi(T)}$$
$$=\sup_{r\in[0,1]}\sqrt{(1-r)\bar{\sigma}_1^2+r\bar{\sigma}_2^2}=\bar{\sigma}_1\vee\bar{\sigma}_2.$$

□

Lemma 5 *Consider* $g(t), h(t)\in\mathbb{D}$*, and a Lipschitz continuous function* f *on* $[0,\infty)$ *with Lipschitz constant* $c>0$*. If* $\sup\limits_{0\le t\le T}|h(t)|=o(\varphi(T))$*, then*

$$\limsup_{T\to\infty}\frac{\|f\circ(g+h)\|_T}{\varphi(T)}=\limsup_{T\to\infty}\frac{\|f\circ g\|_T}{\varphi(T)}.$$

Proof We have

$$\begin{aligned}\left|\|f\circ(g+h)\|_T-\|f\circ g\|_T\right| &\le \|f\circ(g+h)-f\circ g\|_T\\ &=\left|\sup_{0\le t\le T} f(g+h)(t)-\sup_{0\le t\le T} f(g)(t)\right|\\ &\le c\sup_{0\le t\le T}|h(t)|,\end{aligned}$$

under the condition $\sup_{0\le t\le T}h(t)=o(\varphi(T))$. □

Lemmas 3 and 4 can be viewed as generalized versions of the LIL for the BM in (2). All three Lemmas will be used to prove the LILs for the CL and type-1 OL cases (i.e., Theorems 3 and 4).

Proof of Theorem 3 Recall that $\bar{Q}_k(t)=\bar{Z}_k(t)=0$, $\bar{B}_k(t)=\rho_k t$ for $k=1,2,\ldots,K$. And by Corollary 3, we have

$$\left\|\sum_{l=1}^{K-1}\frac{\mu_K}{\mu_l}\widetilde{Q}_l\right\|_T=o(\varphi(T)),\quad \text{w.p.1.} \tag{37}$$

According to (29), we have $\widetilde{Q}_K(t)=\Phi(\widetilde{X}_K)(t)$ and $\widetilde{Y}_K(t)=\Psi(\widetilde{X}_K)(t)$, where

$$\widetilde{X}_K(t)=-\sum_{l=1}^{K-1}\frac{\mu_K}{\mu_l}\widetilde{Q}_l(t)+W_K(t). \tag{38}$$

□

LIL for idle process I_K. Lemma 5 and (38) imply that

$$\limsup_{T\to\infty}\frac{\sup\limits_{0\le t\le T}\widetilde{Y}_K(t)}{\varphi(T)}=\limsup_{T\to\infty}\frac{\sup\limits_{0\le t\le T}\Psi(W_K(t))}{\varphi(T)}$$
$$=\limsup_{T\to\infty}\frac{\sup\limits_{0\le t\le T}W_K(t)}{\varphi(T)}=\mu_K\sigma_K,\quad \text{w.p.1.,} \tag{39}$$

where the second equality holds because $W_K(0)=0$ so that $\Psi(W_k)(t)=\sup\limits_{0\le t\le T}[-W_K(t)]$. Hence, the LIL for I_K follows from (27) and (39).

LILs for queue length Q_K *and workload* Z_K. Combining (27), (38), and Lemma 5 yields that

$$\limsup_{T\to\infty}\frac{\sup\limits_{0\le t\le T}\widetilde{Q}_K(t)}{\varphi(T)}=\limsup_{T\to\infty}\frac{\sup\limits_{0\le t\le T}\Phi(W_K)(t)}{\varphi(T)}$$
$$=\limsup_{T\to\infty}\frac{\sup\limits_{0\le t\le T}\left\{W_K(t)+\sup\limits_{0\le s\le t}[-W_K(s)]\right\}}{\varphi(T)}$$
$$=\limsup_{T\to\infty}\frac{\sup\limits_{0\le t\le T}\sup\limits_{0\le s\le t}W_K(t-s)}{\varphi(T)}$$
$$=\limsup_{T\to\infty}\frac{\sup\limits_{0\le t\le T}W_K(t)}{\varphi(T)}=\mu_K\sigma_K,\quad \text{w.p.1.,}$$

which proves the LIL for Q_K. This and (27) furthermore prove the LIL for Z_K with $Z_K^*=\sigma_K$.

LIL for busy time B_K. First, (29) and (30) imply that

$$\bar{B}_K(t)-\widetilde{B}_K(t)=\frac{1}{\mu_K}[\widetilde{X}_K(t)+\widetilde{Y}_K(t)]-\frac{1}{\mu_K}\widetilde{W}_K(t)$$
$$=-\sum_{l=1}^{K-1}\frac{1}{\mu_l}\widetilde{Q}_l(t)+\sum_{l=1}^{K-1}\frac{1}{\mu_l}\widetilde{W}_l(t)+\frac{1}{\mu_K}\widetilde{Y}_K(t). \tag{40}$$

We then have

$$\limsup_{T\to\infty}\frac{\left\|\bar{B}_K-\widetilde{B}_K\right\|_T}{\varphi(T)}=\frac{1}{\mu_K}\limsup_{T\to\infty}\frac{\left\|\sum_{l=1}^{K-1}\frac{\mu_K}{\mu_l}\widetilde{W}_l+\widetilde{Y}_K\right\|_T}{\varphi(T)}$$

$$= \limsup_{T\to\infty} \frac{\sup_{0\le t\le T}\left|\sup_{0\le s\le t}\left[\sum_{l=1}^{K-1}\frac{\mu_K}{\mu_l}\widetilde{W}_l(t) - W_K(s)\right]\right|}{\mu_K\,\varphi(T)}, \tag{41}$$

where the first equality holds by (40) and Corollary 3, and the second equality holds because, by (38),

$$\widetilde{Y}_K(t) = \sup_{0\le s\le t}\left[-\widetilde{X}_K(s)\right] = \sup_{0\le s\le t}\left[\sum_{l=1}^{K-1}\frac{\mu_K}{\mu_l}\widetilde{Q}_l(s) - W_K(s)\right]. \tag{42}$$

Note that

$$\sum_{l=1}^{K-1}\frac{\mu_K}{\mu_l}\widetilde{W}_l(t) - W_K(s) \overset{\mathrm{d}}{=} \sum_{l=1}^{K-1}\frac{\mu_K}{\mu_l}\widetilde{W}_l(t-s) - \widetilde{W}_K(s),$$

is the sum of two independent BMs with variances $\mu_K^2\sigma_{K-1}^2$ and $\lambda_K c_K^2$. We exploit Lemma 4 to complete the proof of the LIL for B_K.

LIL for departure process D_K. By (13), (27), and (30), we have

$$\begin{aligned}
&\widetilde{D}_K(t) - \bar{D}_K(t)\\
&\quad = -\widetilde{Q}_K(t) + \lambda_K^{1/2}c_{a,K}W_{a,K}(t)\\
&\quad = -\widetilde{X}_K(t) - \widetilde{Y}_K(t) + \lambda_K^{1/2}c_{a,K}W_{a,K}(t)\\
&\quad = \sum_{l=1}^{K-1}\frac{\mu_K}{\mu_l}\widetilde{Q}_l(t) - W_K(t) - \widetilde{Y}_K(t) + \lambda_K^{1/2}c_{a,K}W_{a,K}(t)\\
&\quad = -\left[-\sum_{l=1}^{K-1}\frac{\mu_K}{\mu_l}\widetilde{Q}_l(t) + \sum_{l=1}^{K-1}\frac{\mu_K}{\mu_l}\widetilde{W}_k(t) - \mu_K^{1/2}c_{s,K}W_{s,K}(\bar{B}_K(t)) + \widetilde{Y}_K(t)\right],
\end{aligned}$$

where the second equality follows from (27) and the third equality follows from (38). The rest of the proof is similar to that of B_K with

$$-\sum_{l=1}^{K-1}\frac{\mu_K}{\mu_l}\widetilde{Q}_l(t) + \sum_{l=1}^{K-1}\frac{\mu_K}{\mu_l}\widetilde{W}_l(t) - \mu_K^{1/2}c_{s,K}W_{s,K}(\bar{B}_K(t))$$

replacing the function

$$-\sum_{l=1}^{K-1}\frac{1}{\mu_l}\widetilde{Q}_l(t) + \sum_{l=1}^{K-1}\frac{1}{\mu_l}\widetilde{W}_l(t)$$

in (40).

6.4 Proof of Theorem 4

We now consider a type-1 OL $(GI/GI)^K/1$/PPSD model with $\rho > 1$ and $\sum_{k=1}^{k_0} \rho_k = 1$ for $1 \le k_0 < K$. Because the first k_0 classes naturally form a CL $(GI/GI)^{k_0}/1$/PPSD model (already discussed in Theorem 3 and proved in Sect. 6.3), it remains to prove the LILs for classes $k_0+1, k_0+2, \ldots, K$. We seek help from Lemmas 3–4 and the following Lemma 6.

Lemma 6 *Consider a type-1 OL system defined in (19). For* $i = 1, \ldots, K - k_0$,

$$\widetilde{X}_{k_0+i}(t) - \bar{X}_{k_0+i}(t) = \widetilde{W}_{k_0+i}(t) - \mu_{k_0+i}\widetilde{I}_{k_0+i-1}(t), \tag{43}$$

$$\widetilde{Q}_{k_0+i}(t) - \bar{Q}_{k_0+i}(t) = \widetilde{W}_{k_0+i}(t) - \mu_{k_0+i}\left[\widetilde{I}_{k_0+i-1}(t) - \widetilde{I}_{k_0+i}(t)\right]. \tag{44}$$

Proof Consider a type-1 OL $(GI/GI)^K/1$/PPSD system, we have $\bar{I}_k(t) = 0$ for all t and $k = k_0, k_0+1, \ldots, K$. First, (29) implies that

$$\widetilde{X}_{k_0}(t) - \bar{X}_{k_0}(t) = -\sum_{l=1}^{k_0-1} \frac{\mu_{k_0}}{\mu_l}\left[\widetilde{Q}_l(t) - \bar{Q}_l(t)\right] + W_{k_0}(t). \tag{45}$$

In addition, since $\bar{Q}_{k_0}(t) = \bar{X}_{k_0}(t) = 0$, we have

$$\begin{aligned}\widetilde{Q}_{k_0}(t) - \bar{Q}_{k_0}(t) &= \widetilde{X}_{k_0}(t) + \widetilde{Y}_{k_0}(t)\\ &= -\sum_{l=1}^{k_0-1} \frac{\mu_{k_0}}{\mu_l}\left[\widetilde{Q}_l(t) - \bar{Q}_l(t)\right] + W_{k_0}(t) + \widetilde{Y}_{k_0}(t).\end{aligned} \tag{46}$$

Following (29), we have

$$\widetilde{X}_{k_0+1}(t) - \bar{X}_{k_0+1}(t) = -\sum_{l=1}^{k_0} \frac{\mu_{k_0+1}}{\mu_l}\left[\widetilde{Q}_l(t) - \bar{Q}_l(t)\right] + W_{k_0+1}(t). \tag{47}$$

Substituting (46) into (47) yields (43) for $i = 1$, and then (44) for $i = 1$ with the help of (27).

We next prove (43) and (44) using induction. Assume (43) holds for $1, 2, \ldots, i$ with $i < K - k_0$, (29) implies that

$$\begin{aligned}&\widetilde{X}_{k_0+i+1}(t) - \bar{X}_{k_0+i+1}(t)\\ &= -\sum_{l=1}^{k_0} \frac{\mu_{k_0+i+1}}{\mu_l}\left[\widetilde{Q}_l(t) - \bar{Q}_l(t)\right] - \sum_{l=k_0+1}^{k_0+i} \frac{\mu_{k_0+i+1}}{\mu_l}\left[\widetilde{Q}_l(t) - \bar{Q}_l(t)\right] + W_{k_0+i+1}(t)\\ &= W_{k_0+i+1}(t) - \frac{\mu_{k_0+i+1}}{\mu_{k_0}}\left[W_{k_0}(t) + \widetilde{Y}_{k_0}(t)\right]\\ &\quad - \sum_{l=k_0+1}^{k_0+i} \frac{\mu_{k_0+i+1}}{\mu_l}\left\{\widetilde{W}_l(t) - \mu_l\left[\widetilde{I}_{l-1}(t) - \widetilde{I}_l(t)\right]\right\}\end{aligned}$$

$$= W_{k_0+i+1}(t) - \frac{\mu_{k_0+i+1}}{\mu_{k_0}} W_{k_0}(t) - \sum_{l=k_0+1}^{k_0+i} \frac{\mu_{k_0+i+1}}{\mu_l} \widetilde{W}_l(t) - \mu_{k_0+i+1} \widetilde{I}_{k_0+i}(t)$$
$$= \widetilde{W}_{k_0+i+1} - \mu_{k_0+i+1} \widetilde{I}_{k_0+i}(t),$$

where the second equality holds by (46) and the induction hypothesis. Hence, (43) holds for $i+1$. □

6.4.1 LIL for class k_0+1

We first prove the LIL for class k_0+1.

LIL for idle process I_{k_0+1}. Theorem 1 (also see Sect. 1) implies that $\bar{X}_k(t) = \lambda_k t$ for $k \geq k_0+1$, $\bar{B}_k(t) = \rho_k t$ for $1 \leq k \leq k_0$, and $\bar{B}_k(t) = 0$ for $k \geq k_0+1$. Following (29), we have

$$\widetilde{X}_{k_0+1}(t) = \bar{X}_{k_0+1}(t) - \sum_{l=1}^{k_0} \frac{\mu_{k_0+1}}{\mu_l} \widetilde{Q}_l(t) + \sum_{l=1}^{k_0} \frac{\mu_{k_0+1}}{\mu_l} \widetilde{W}_l(t)$$
$$+ \lambda_{k_0+1}^{1/2} c_{a,k_0+1} W_{a,k_0+1}(t). \tag{48}$$

Observe that the last two terms in (48) are two driftless BMs. Because LILs hold for the first k_0 classes with limits $Q_k^* = 0$ for $k = 1, 2, \ldots, k_0 - 1$ and $Q_{k_0}^* = \mu_{k_0}\sigma_{k_0}$, we have $\lim_{t\to\infty} \widetilde{X}_{k_0+1}(t)/t = \lambda_{k_0+1}$ or equivalently, $\lim_{t\to\infty} \widetilde{X}_{k_0+1}(t) = \infty$, w.p.1. Therefore, by (27) and the definition of the ORM Ψ in (14), $\sup_{t\geq 0} \widetilde{Y}_{k_0+1}(t) < \infty$ w.p.1. so that

$$\limsup_{T\to\infty} \frac{\|\widetilde{Y}_{k_0+1}\|_T}{\varphi(T)} = 0, \quad \text{w.p.1.}$$

This, together with (27), proves the LIL for I_{k_0+1} with $I_{k_0+1}^* = 0$.

LIL for queue length Q_{k_0+1}. Note $\bar{Q}_k(t) = 0$ for all $k = 1, 2, \ldots, k_0$ under $\sum_{l=1}^{k_0} \rho_l = 1$. Hence, (14) and (29) imply that

$$\widetilde{I}_{k_0}(t) = \frac{\sup_{0\leq s\leq t} \left[-\widetilde{X}_{k_0}(s)\right]}{\mu_{k_0}} = \sup_{0\leq s\leq t} \left[\sum_{l=1}^{k_0-1} \frac{\widetilde{Q}_l(s)}{\mu_l} - \frac{W_{k_0}(s)}{\mu_{k_0}}\right]. \tag{49}$$

Following (44), we have

$$\limsup_{T\to\infty} \frac{\|\widetilde{Q}_{k_0+1} - \bar{Q}_{k_0+1}\|_T}{\varphi(T)}$$
$$= \limsup_{T\to\infty} \frac{\left\|\lambda_{k_0+1}^{1/2} c_{a,k_0+1} W_{a,k_0+1} - \mu_{k_0+1}\left(\widetilde{I}_{k_0} - \widetilde{I}_{k_0+1}\right)\right\|_T}{\varphi(T)}$$

$$
\begin{aligned}
&= \limsup_{T\to\infty} \frac{\left\| \lambda_{k_0+1}^{1/2} c_{a,k_0+1} W_{a,k_0+1} - \mu_{k_0+1} \widetilde{I}_{k_0} \right\|_T}{\varphi(T)}, \\
&= \limsup_{T\to\infty} \frac{\sup_{0\le t\le T} \left| \lambda_{k_0+1}^{1/2} c_{a,k_0+1} W_{a,k_0+1}(t) + \sup_{0\le s\le t} \frac{\mu_{k_0+1}}{\mu_{k_0}} W_{k_0}(s) \right|}{\varphi(T)} \\
&= \sqrt{\lambda_{k_0+1} c_{a,k_0+1}^2 + \mu_{k_0+1}^2 \sigma_{k_0}^2},
\end{aligned} \tag{50}
$$

where the second equality holds because $I_{k_0+1}^* = 0$, the third equality holds because, by (27) and (29),

$$
\begin{aligned}
&\left\| \lambda_{k_0+1}^{1/2} c_{a,k_0+1} W_{a,k_0+1} - \mu_{k_0+1} \widetilde{I}_{k_0} \right\|_T \\
&\quad = \sup_{0\le t\le T} \left| \sup_{0\le s\le t} \left\{ -\lambda_{k_0+1}^{1/2} c_{a,k_0+1} W_{a,k_0+1}(t) + \left[\sum_{l=1}^{k_0-1} \frac{\mu_{k_0+1}}{\mu_l} \widetilde{Q}_l(s) - \frac{\mu_{k_0+1}}{\mu_{k_0}} W_{k_0}(s) \right] \right\} \right|,
\end{aligned}
$$

and $Q_k^* = 0, k = 1, 2, \ldots, k_0 - 1$, and the last equality in (44) holds by Lemma 3 with two independent BMs $W_{a,k_0+1}(t)$ and $W_{k_0}(t)$ having variance parameters 1 and $\mu_{k_0}\sigma_{k_0}$.

LIL for workload process Z_{k_0+1}. Because $\bar{B}_{k_0+1}(t) = 0$, it follows from (13), (27), and (44) that

$$
\begin{aligned}
&\widetilde{Z}_{k_0+1}(t) - \bar{Z}_{k_0+1}(t) \\
&\quad = \frac{1}{\mu_{k_0+1}} \left[\widetilde{Q}_{k_0+1}(t) - \mu_{k_0+1}^{1/2} c_{s,k_0+1} W_{s,k_0+1}(\rho_{k_0+1} t) \right] - \frac{1}{\mu_{k_0+1}} \bar{Q}_{k_0+1}(t) \\
&\quad = \frac{1}{\mu_{k_0+1}} \widetilde{W}_{k_0+1}^*(t) - \left[\widetilde{I}_{k_0}(t) - \widetilde{I}_{k_0+1}(t) \right],
\end{aligned}
$$

where $\widetilde{W}_{k_0+1}^*(t) \equiv \lambda_{k_0+1}^{1/2} c_{a,k_0+1} W_{a,k_0+1}(t) - \mu_{k_0+1}^{1/2} c_{s,k_0+1} W_{s,k_0+1}(\rho_{k_0+1} t)$. Hence, with $I_{k_0+1}^* = 0$, we have

$$
\begin{aligned}
&\limsup_{T\to\infty} \frac{\left\| \widetilde{Z}_{k_0+1} - \bar{Z}_{k_0+1} \right\|_T}{\varphi(T)} \\
&\quad = \frac{1}{\mu_{k_0+1}} \limsup_{T\to\infty} \frac{\sup_{0\le t\le T} \left| -\mu_{k_0+1} \widetilde{I}_{k_0}(t) + \widetilde{W}_{k_0+1}^*(t) \right|}{\varphi(T)} \\
&\quad = \frac{1}{\mu_{k_0+1}} \limsup_{T\to\infty} \frac{\sup_{0\le t\le T} \left| \sup_{0\le s\le t} \frac{\mu_{k_0+1}}{\mu_{k_0}} W_{k_0}(s) + \widetilde{W}_{k_0+1}^*(t) \right|}{\varphi(T)} = Z_{k_0+1}^*, \quad \text{w.p.1.},
\end{aligned}
$$

where the second equality holds by similar analysis for Q_{k_0+1} in (50) and the third equality holds by Lemma 3 with two independent BMs W_{k_0} and $\widetilde{W}^*_{k_0+1}$ with variance parameters $\mu_{k_0}\sigma_{k_0}$ and $\sqrt{\lambda_{k_0+1}}c_{k_0+1}$.

LIL for busy time B_{k_0+1} *and departure process* D_{k_0+1}. Because $\widetilde{B}_k(t) = \widetilde{I}_{k-1}(t) - \widetilde{I}_k(t)$ for $k = 2, 3, \ldots, K$ and $I^*_{k_0+1} = 0$, (27) implies the LIL for B_{k_0+1}, that is,

$$\limsup_{T\to\infty} \frac{\|\widetilde{B}_{k_0+1}\|_T}{\varphi(T)} = \limsup_{T\to\infty} \frac{\|\widetilde{I}_{k_0}\|_T}{\varphi(T)} = \sigma_{k_0}, \quad \text{w.p.1.}$$

Finally, according to (27) and (13) (also see (56)), we have $\widetilde{D}_{k_0+1} = \mu_{k_0+1}\,\widetilde{B}_{k_0+1}$ and $\bar{D}_{k_0+1} = \bar{B}_{k_0+1} = 0$, which concludes the LIL for D_{k_0+1}.

6.4.2 LIL for classes $k_0 + 2$ to K

We next prove the LIL for classes $k_0 + 2$ to K.

LIL for idle processes $I_{k_0+2}, \ldots, I_K$. We prove the LIL for I_k using induction. First, since $\bar{B}_{k_0+2}(t) = 0$, rewriting (43) yields that

$$\widetilde{X}_{k_0+2}(t) - \bar{X}_{k_0+2}(t) = \lambda^{1/2}_{k_0+2}c_{a,k_0+2}W_{a,k_0+2}(t) - \mu_{k_0+2}\widetilde{I}_{k_0+1}(t).$$

Because $\bar{X}_{k_0+2}(t) = \lambda_{k_0+2}t$, $I^*_{k_0+1} = 0$, we obtain the LIL for I_{k_0+2} with $I^*_{k_0+2} = 0$, following the analysis for the LIL of I_{k_0+1} in Sect. 6.4.1.

Next, suppose the LIL holds for I_k with $I^*_k = 0$ for $k = k_0 + 2, k_0 + 3, \ldots, k_0 + i$ with $i < K - k_0$. Since $\bar{B}_{k_0+i+1}(t) = 0$, (43) implies that

$$\widetilde{X}_{k_0+i+1}(t) - \bar{X}_{k_0+i+1}(t) = \lambda^{1/2}_{k_0+i+1}c_{a,k_0+i+1}W_{a,k_0+i+1}(t) - \mu_{k_0+i+1}\widetilde{I}_{k_0+i}(t).$$

We have $\bar{X}_{k_0+i+1}(t) = \lambda_{k_0+i+1}t$ and $\widetilde{I}^*_{k_0+i} = 0$ following the induction hypothesis. We then establish the LIL for I_{k_0+i+1} with $I^*_{k_0+i+1} = 0$, using the similar analysis of the LIL of I_{k_0+2}.

LIL for queue lengths $Q_{k_0+2}, \ldots, Q_K$. Since $\bar{B}_k(t) = 0, k \geq k_0 + 2$, by (44), we have,

$$\widetilde{Q}_k(t) - \bar{Q}_k(t) = \lambda^{1/2}_k c_{a,k}W_{a,k}(t) - \mu_k\left[\widetilde{I}_{k-1}(t) - \widetilde{I}_k(t)\right] \quad \text{for} \quad k \geq k_0 + 2.$$

Because $I^*_k = 0$ for all $k = k_0 + 1, k_0 + 2, \ldots, K$, we have

$$\limsup_{T\to\infty} \frac{\|\widetilde{Q}_k - \bar{Q}_k\|_T}{\varphi(T)} = \limsup_{T\to\infty} \frac{\left\|\lambda^{1/2}_k c_{a,k}W_{a,k}\right\|_T}{\varphi(T)} = \sqrt{\lambda_k}c_{a,k}, \quad \text{w.p.1.}$$

LIL for workload processes $Z_{k_0+2}, \ldots, Z_K$. Since $\bar{B}_k(t) = 0$, $k \geq k_0 + 2$, according to (27) and (44), we have, for $k = k_0 + 2, k_0 + 3, \ldots, K$,

$$\begin{aligned}
&\widetilde{Z}_k(t) - \bar{Z}_k(t) \\
&= \frac{1}{\mu_k}\left[\widetilde{Q}_k(t) - \mu_k^{1/2} c_{s,k} W_{s,k}(\rho_k t)\right] - \frac{1}{\mu_k}\bar{Q}_k(t) \\
&= \frac{1}{\mu_k}\left[\lambda_k^{1/2} c_{a,k} W_{a,k}(t) - \mu_k^{1/2} c_{s,k} W_{s,k}(\rho_k t)\right] - \left[\widetilde{I}_{k-1}(t) - \widetilde{I}_k(t)\right].
\end{aligned}$$

Because $I_k^* = 0$ for all $k = k_0 + 1, k_0 + 2, \ldots, K$, we have,

$$\limsup_{T\to\infty} \frac{\|\widetilde{Z}_k - \bar{Z}_k\|_T}{\varphi(T)} = \frac{c_k\sqrt{\lambda_k}}{\mu_k}, \quad \text{w.p.1.,} \quad k = k_0 + 2, k_0 + 3, \ldots, K.$$

LIL for busy times $B_{k_0+2}, \ldots, B_K$ *and departure* $D_{k_0+2}, \ldots, D_K$. Because $\widetilde{B}_k(t) = \widetilde{I}_{k-1}(t) - \widetilde{I}_k(t)$ and $\widetilde{D}_k(t) = \mu_k \widetilde{B}_k(t)$ (due to (27)), the LILs for B_k and D_k with $B_k^* = D_k^* = 0$ follow from the LILs for I_k and I_{k-1}, $k = k_0 + 2, k_0 + 3, \ldots, K$,

6.5 Proofs of Theorems 5

Consider a type-2 OL $(GI/GI)^K/1/\text{PPSD}$ model with $\rho > 1$, $\sum_{k=1}^{k_0} \rho_k < 1$ and $\sum_{k=1}^{k_0+1} \rho_k > 1$ for $1 \leq k_0 < K$. The first k_0 classes form a UL $(GI/GI)^{k_0}/1/\text{PPSD}$ system (already discussed in Theorem 2 and proved in Sect. 6.2). In addition, the treatment of the LIL for classes $k_0 + 2$ to K is analogous to that of the type-1 OL model (discussed in Sect. 6.4). Therefore, it remains to prove the LILs for classes $k_0 + 1$.

LIL for idle time I_{k_0+1}. Following (29), we have

$$\widetilde{X}_{k_0+1}(t) = \bar{X}_{k_0+1}(t) - \sum_{l=1}^{k_0} \frac{\mu_{k_0+1}}{\mu_l}\widetilde{Q}_l(t) + \sum_{l=1}^{k_0+1} \frac{\mu_{k_0+1}}{\mu_l}\widetilde{W}_l(t), \tag{51}$$

where $\bar{X}_{k_0+1}(t) = \mu_{k_0+1}\left(\sum_{l=1}^{k_0+1} \rho_l - 1\right) t$ by (13). Because $\widetilde{Q}_j(t) = O(\log(t))$ for $1 \leq j \leq k_0$ (by Corollary 3) so that the second term in (51) is $o(t)$ and the third term in (51) is a driftless BM, we have $\lim_{t\to\infty} \widetilde{X}_{k_0+1}(t)/t = \mu_{k_0+1}(\sum_{l=1}^{k_0+1} \rho_l - 1) > 0$ w.p.1. Therefore, the LIL for I_{k_0+1} holds with $I_{k_0+1}^* = 0$ following similar analysis of the proof for the LIL of I_{k_0+2} in Theorem 4.

LIL for queue length Q_{k_0+1}. We write

$$\limsup_{T\to\infty} \frac{\sup_{0\leq t\leq T} \left|\widetilde{Q}_{k_0+1}(t) - \bar{Q}_{k_0+1}(t)\right|}{\varphi(T)}$$

$$= \limsup_{T\to\infty} \frac{\sup_{0\le t\le T} \left|\widetilde{X}_{k_0+1}(t) + \widetilde{Y}_{k_0+1}(t) - \bar{X}_{k_0+1}(t)\right|}{\varphi(T)}$$

$$= \limsup_{T\to\infty} \frac{\sup_{0\le t\le T} \left|\widetilde{X}_{k_0+1}(t) - \bar{X}_{k_0+1}(t)\right|}{\varphi(T)} = \limsup_{T\to\infty} \frac{\sup_{0\le t\le T} \left|\sum_{l=1}^{k_0+1} \frac{\mu_{k_0+1}}{\mu_l} \widetilde{W}_l(t)\right|}{\varphi(T)} = Q^*_{k_0+1},$$

where $Q^*_{k_0+1}$ is given in (24), the first equality holds because $\bar{Q}_{k_0+1}(t) = \bar{X}_{k_0+1}(t)$, the second equality holds because $I^*_{k_0+1} = 0$, the third equality holds by (51) and Corollary 3, and the last equality holds by (2) and $\bar{B}_k(t) = \rho_k t$, $k = 1, 2, \ldots, k_0$ and $\bar{B}_{k_0+1}(t) = (1 - \sum_{i=1}^{k_0} \rho_i)t$.

LIL for busy time B_{k_0+1}. Following (13) and (27), we have

$$\begin{aligned}\bar{B}_{k_0+1}(t) - \widetilde{B}_{k_0+1}(t) &= \left[\bar{I}_{k_0}(t) - \bar{I}_{k_0+1}(t)\right] - \left[\widetilde{I}_{k_0}(t) - \widetilde{I}_{k_0+1}(t)\right] \\ &= \left[\bar{I}_{k_0}(t) - \widetilde{I}_{k_0}(t)\right] - \left[\bar{I}_{k_0+1}(t) - \widetilde{I}_{k_0+1}(t)\right].\end{aligned}$$

The LIL for B_{k_0+1} thus holds with $B^*_{k_0+1} = \sigma_{k_0}$ because $I^*_{k_0} = \sigma_{k_0}$ and $I^*_{k_0+1} = 0$.

LIL for workload process Z_{k_0+1}. Note that $\bar{Q}_{k_0+1}(t) = \bar{X}_{k_0+1}(t)$ and $\widetilde{Q}_{k_0+1}(t) = \widetilde{X}_{k_0+1}(t) + \widetilde{Y}_{k_0+1}(t)$, so (27) and (29) imply

$$\begin{aligned}&\widetilde{Z}_{k_0+1}(t) - \bar{Z}_{k_0+1}(t) \\ &\quad= \frac{1}{\mu_{k_0+1}} \left[\widetilde{X}_{k_0+1}(t) - \bar{X}_{k_0+1}(t)\right] + \widetilde{I}_{k_0+1}(t) \\ &\quad\quad+ \frac{1}{\mu_{k_0+1}} \left[\mu^{1/2}_{k_0+1} c_{s,k_0+1} W_{s,k_0+1}(\bar{B}_{k_0+1}(t)) - \mu^{1/2}_{k_0+1} c_{s,k_0+1} W_{s,k_0+1}(\rho_{k_0+1} t)\right] \\ &\quad= \widetilde{I}_{k_0+1}(t) - \sum_{l=1}^{k_0} \frac{1}{\mu_l} \widetilde{Q}_l(t) + \sum_{l=1}^{k_0+1} \frac{1}{\mu_l} \left[\lambda_l^{1/2} c_{a,l} W_{a,l}(t) - \mu_l^{1/2} c_{s,l} W_{s,l}(\rho_l t)\right],\end{aligned}$$

where the second equality holds by (51). The LIL for I_{k_0+1} (with $I^*_{k_0+1} = 0$) and Corollary 3 imply that

$$\begin{aligned}&\limsup_{T\to\infty} \frac{\sup_{0\le t\le T} \left|\widetilde{Z}_{k_0+1}(t) - \bar{Z}_{k_0+1}(t)\right|}{\varphi(T)} \\ &\quad= \limsup_{T\to\infty} \frac{\sup_{0\le t\le T} \left|\sum_{l=1}^{k_0+1} \frac{1}{\mu_l} \left[\lambda_l^{1/2} c_{a,l} W_{a,l}(t) - \mu_l^{1/2} c_{s,l} W_{s,l}(\rho_l t)\right]\right|}{\varphi(T)} = \sigma_{k_0+1}.\end{aligned}$$

LIL for departure process D_{k_0+1}. Following (13), (27), (29), and (30), we have

$$\begin{aligned}
&\widetilde{D}_{k_0+1}(t) - \bar{D}_{k_0+1}(t)\\
&\quad = \mu_{k_0+1}\left[\widetilde{B}_{k_0+1}(t) - \bar{B}_{k_0+1}(t)\right] + \mu_{k_0+1}^{1/2} c_{s,k_0+1} W_{s,k_0+1}(\bar{B}_{k_0+1}(t))\\
&\quad = \left[\bar{Q}_{k_0+1}(t) - \widetilde{Q}_{k_0+1}(t)\right] + \lambda_{k_0+1}^{1/2} c_{a,k_0+1} W_{a,k_0+1}(t)\\
&\quad = \left[\bar{X}_{k_0+1}(t) - \widetilde{X}_{k_0+1}(t)\right] - \widetilde{Y}_{k_0+1}(t) + \lambda_{k_0+1}^{1/2} c_{a,k_0+1} W_{a,k_0+1}(t)\\
&\quad = \sum_{l=1}^{k_0} \frac{\mu_{k_0+1}}{\mu_l} \widetilde{Q}_l(t) - \widetilde{Y}_{k_0+1}(t) - \sum_{l=1}^{k_0} \frac{\mu_{k_0+1}}{\mu_l} \widetilde{W}_l(t)\\
&\qquad + \mu_{k_0+1}^{1/2} c_{s,k_0+1} W_{s,k_0+1}(\bar{B}_{k_0+1}(t)),
\end{aligned}$$

where the third equality holds because $\bar{Q}_{k_0+1}(t) = \bar{X}_{k_0+1}(t)$ and the last equality follows from (51).

Since $Q_k^* = 0$ for all $k = 1, 2, \ldots, k_0$ and $I_{k_0+1}^* = 0$, it follows that

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{\sup_{0\le t\le T} \left|\widetilde{D}_{k_0+1}(t) - \bar{D}_{k_0+1}(t)\right|}{\varphi(T)}\\
&= \limsup_{T\to\infty} \frac{\sup_{0\le t\le T} \left| -\sum_{l=1}^{k_0} \frac{\mu_{k_0+1}}{\mu_l} \widetilde{W}_l(t) + \mu_{k_0+1}^{1/2} c_{s,k_0+1} W_{s,k_0+1}(\bar{B}_{k_0+1}(t)) \right|}{\varphi(T)} = D_{k_0+1}^*,
\end{aligned}$$

where $D_{k_0+1}^*$ is given in (24), the second equality holds by Corollary 3 and the LIL for I_{k_0+1} (with $I_{k_0+1}^* = 0$), and the last equality holds because $\bar{B}_l(t) = \rho_l t$ for $l = 1, 2, \ldots, k_0$ and $\bar{B}_{k_0+1}(t) = (1 - \sum_{l=1}^{k_0} \rho_l)t$, so that

$$-\sum_{l=1}^{k_0} \frac{\mu_{k_0+1}}{\mu_l} \widetilde{W}_l(t) + \mu_{k_0+1}^{1/2} c_{s,k_0+1} W_{s,k_0+1}(\bar{B}_{k_0+1}(t))$$

is a driftless BM.

6.6 Proof of Theorem 6

Consider a $(GI/GI)^K/1/\text{PPSD}$ model with $\rho_1 > 1$. The LILs for classes 2 to K are similar to those of classes $k_0 + 2$ to K in a type-2 OL $(GI/GI)^K/1/\text{PPSD}$ system, discussed in Theorem 5 and proved in Sect. 6.5. Hence, it remains to treat the first class.

Because $\rho_1 > 1$, $\bar{X}_1(t) = (\lambda_1 - \mu_1)t > 0$, (29) implies that

$$\widetilde{X}_1(t) = \bar{X}_1(t) + \lambda_1^{1/2} c_{a,1} W_{a,1}(t) - \mu_1^{1/2} c_{s,1} W_{s,1}(t),$$

that is a BM with positive drift $\lambda_1 - \mu_1 > 0$. Therefore, similar to the analysis in Sect. 6.5, we have $\lim_{t\to\infty} \widetilde{X}_1(t) = \infty$ and $\sup_{t\geq 0} I(t) < \infty$, w.p.1., which concludes the LILs for I_1 and B_1 with $I_1^* = 0$ and $B_1^* = 0$ because $\widetilde{I}_1(t) - \bar{I}_1(t) = \bar{B}_1(t) - \widetilde{B}_1(t)$.

Next, (13) and (27) imply that $\bar{Q}_1(t) = \bar{X}_1(t)$ and $\widetilde{Q}_1(t) = \widetilde{X}_1(t) + \mu_1 \widetilde{I}_1(t)$, so that

$$\limsup_{T\to\infty} \frac{\|\widetilde{Q}_1 - \bar{Q}_1\|_T}{\varphi(T)} = \limsup_{T\to\infty} \frac{\left\|\lambda_1^{1/2} c_{a,1} W_{a,1} - \mu_1^{1/2} c_{s,1} W_{s,1}\right\|_T}{\varphi(T)}$$
$$= \sqrt{\lambda_1 c_{a,1}^2 + \mu_1 c_{s,1}^2}., \quad \text{w.p.1.}$$

Regarding the LIL for Z_1, (13) and (27) imply that

$$\bar{Z}_1(t) = \frac{\bar{Q}_1(t)}{\mu_1} \quad \text{and} \quad \widetilde{Z}_1(t) = \frac{\widetilde{Q}_1(t)}{\mu_1} + \frac{\mu_1^{1/2} c_{s,1} W_{s,1}(t) - \mu_1^{1/2} c_{s,1} W_{s,1}(\rho_1 t)}{\mu_1},$$

so that

$$\begin{aligned}
\widetilde{Z}_1(t) - \bar{Z}_1(t) &= \frac{1}{\mu_1}\left[\widetilde{Q}_1(t) - \bar{Q}_1(t)\right] + \frac{1}{\mu_1}\left[\mu_1^{1/2} c_{s,1} W_{s,1}(t) - \mu_1^{1/2} c_{s,1} W_{s,1}(\rho_1 t)\right] \\
&= \frac{1}{\mu_1}\left[\lambda_1^{1/2} c_{a,1} W_{a,1}(t) - \mu_1^{1/2} c_{s,1} W_{s,1}(t)\right] + \widetilde{I}_1(t) \\
&\quad + \frac{1}{\mu_1}\left[\mu_1^{1/2} c_{s,1} W_{s,1}(t) - \mu_1^{1/2} c_{s,1} W_{s,1}(\rho_1 t)\right] \\
&= \frac{1}{\mu_1}\left[\lambda_1^{1/2} c_{a,1} W_{a,1}(t) - \mu_1^{1/2} c_{s,1} W_{s,1}(\rho_1 t)\right] + \widetilde{I}_1(t).
\end{aligned}$$

Hence,

$$\limsup_{T\to\infty} \frac{\|\widetilde{Z}_1 - \bar{Z}_1\|_T}{\varphi(T)} = \frac{1}{\mu_1} \limsup_{T\to\infty} \frac{\left\|\lambda_1^{1/2} c_{a,1} W_{a,1} - \lambda_1^{1/2} c_{s,1} W_{s,1}\right\|_T}{\varphi(T)}$$
$$= \frac{1}{\mu_1}\sqrt{\lambda(c_{a,1}^2 + c_{s,1}^2)} = \sigma_1, \quad \text{w.p.1.}$$

Finally, it follows from (13) and (27) that

$$\widetilde{D}_1(t) - \bar{D}_1(t) = \mu_1\left[\widetilde{B}_1(t) - \bar{B}_1(t)\right] + \mu_1^{1/2} c_{s,1} W_{s,1}(\bar{B}_1(t)),$$

where $\bar{B}_1(t) = t$. So, the LIL for D_1 easily follows the LIL for B_1, with $D_1^* = \mu_1^{1/2} c_{s,1}$.

7 Conclusions

We have developed a strong form of LIL for the multiclass $(GI/GI)^K/1/\text{PPSD}$ priority queueing model by focusing on five key performance processes: queue length, workload, busy time, idle time, and departure processes. Refining the FSLLNs and the corresponding limiting fluid functions which are often used to approximate the mean values, the LILs provide an estimate for the asymptotic rate of the increasing stochastic variability of these performance functions. We have identified these LIL limits as explicit functions of the first and second moments of the interarrival and service times of all K classes. Comprehensive discussions and numerical experiments have been provided to gain insights of these LIL limits.

Our main results cover all three important cases UL, CL, and OL categorized by the loading and traffic intensity. There are two important steps in the proofs. First, we have adapted an SA approach to relate the original LIL pre-limit processes to those of the corresponding SAs. Second, we have developed asymptotic theories for functions which involve two BMs. These results are formulated in a series of lemmas which can be viewed the two-dimensional generalization of the conventional LIL for the standard BM (as in (2)), so they are legitimate results in their own right.

Acknowledgments We thank Prof. Ward Whitt, Prof. Junfei Huang and the anonymous referees for providing constructive comments. Both authors were supported by NSFC grant 11471053. The first author also acknowledges support from NSFC grant 11101050. The second author also acknowledges support from NSF Grant CMMI 1362310.

Appendix: Overview

This appendix contains additional materials supplementing the main paper. In Sect. 1 we provide an alternative definition for the one-dimensional ORM in (14). In Sect. 1 we provide the analytic solutions to the fluid Eq. (13). In Sect. 1, more numerical examples are given to supplement Sect. 5. In Sect. 1 we further analyze Corollary 1 in two cases. Finally, in Sect. 1 we prove Lemma 2.

The one-dimensional oblique reflection mapping

We now provide an alternative definition of the ORM in (14).

Definition 1 For any function $x \in \mathbb{D}_0$, if there exists a unique pair of functions $z, y \in \mathbb{D}_0$ satisfying

(i) $z(t) = x(t) + y(t) \geq 0$;
(ii) y is non-decreasing and $y(0) = 0$;
(iii) $\int_0^\infty z(t)\mathrm{d}y(t) = 0$,

is called the one-dimensional *oblique reflection mapping*, denoted by $(z, y) = (\Phi, \Psi)(x)$.

Fluid solution to (13)

We provide analytic solutions to (13) in three cases: (i) $\rho_1 > 1$, (ii) there exists an integer $k_0 : 1 < k_0 < K$ such that $\sum_{l=1}^{k_0} \rho_l \leq 1$ and $\sum_{l=1}^{k_0+1} \rho_l > 1$, and (iii) $\rho < 1$.

Case (i): If $\rho_1 > 1$, then the solution of (13) is

$$\bar{Q}_k(t) = \mu_k \bar{Z}_k(t) = \begin{cases} (\lambda_k - \mu_k)t, & k = 1; \\ \lambda_k t, & k = 2, 3, \ldots, K, \end{cases} \tag{52}$$

$$\bar{B}_k(t) = \begin{cases} t, & k = 1; \\ 0, & k = 2, 3, \ldots, K, \end{cases} \tag{53}$$

$$\bar{I}_k(t) = 0, \qquad k = 1, 2, \ldots, K, \tag{54}$$

and $\bar{X}_k(t) = \bar{Q}_k(t)$ and $\bar{D}_1(t) = \mu_1 t$ and $\bar{D}_k(t) = 0$ for all $k = 2, 3, \ldots, K$.

Case (ii): If there exists an integer $k_0 : 1 < k_0 < K$ such that $\sum_{l=1}^{k_0} \rho_l \leq 1$ and $\sum_{l=1}^{k_0+1} \rho_l > 1$, then the solution of (13) is

$$\begin{aligned} \bar{Q}_k(t) &= \mu_k \bar{Z}_k(t) = \lambda_k t - \left[\rho_k \wedge \left(1 - \sum_{l=1}^{k-1} \rho_l \right)^+ \right] \mu_k t \\ &= \begin{cases} 0, & k = 1, 2, \ldots, k_0; \\ \mu_{k_0+1} \left(\sum_{l=1}^{k_0+1} \rho_l - 1 \right) t, & k = k_0 + 1; \\ \lambda_k t, & k = k_0 + 2, k_0 + 3, \ldots, K, \end{cases} \end{aligned} \tag{55}$$

$$\bar{B}_k(t) = \left[\rho_k \wedge \left(1 - \sum_{l=1}^{k-1} \rho_l \right)^+ \right] t = \begin{cases} \rho_k t, & k = 1, 2, \ldots, k_0; \\ \left(1 - \sum_{l=1}^{k-1} \rho_l \right) t, & k = k_0 + 1; \\ 0, & k = k_0 + 2, k_0 + 3, \ldots, K, \end{cases} \tag{56}$$

$$\bar{I}_k(t) = \left(1 - \sum_{l=1}^{k} \rho_l \right)^+ t = \begin{cases} \left(1 - \sum_{l=1}^{k} \rho_l \right) t, & k = 1, 2, \ldots, k_0; \\ 0, & k = k_0 + 1, k_0 + 2, \ldots, K. \end{cases} \tag{57}$$

Finally, we also note that

$$\bar{X}_k(t) = \mu_k \left[\rho_k t + \sum_{l=1}^{k-1} \bar{B}_l(t) - t \right] = \begin{cases} \mu_k \left(\sum_{l=1}^{k} \rho_l - 1 \right) t, & k = 1, 2, \ldots, k_0 + 1; \\ \lambda_k t, & k = k_0 + 2, k_0 + 3, \ldots, K, \end{cases} \tag{58}$$

$$\bar{D}_k(t) = \mu_k \bar{B}_k(t) = \begin{cases} \lambda_k t, & k = 1, 2, \ldots, k_0; \\ \mu_k \left(1 - \sum_{l=1}^{k-1} \rho_l \right) t, & k = k_0 + 1; \\ 0, & k = k_0 + 2, k_0 + 3, \ldots, K. \end{cases} \tag{59}$$

Case (iii): If $\rho < 1$, all fluid functions satisfy the fluid functions in case (ii) with k_0 replaced by K.

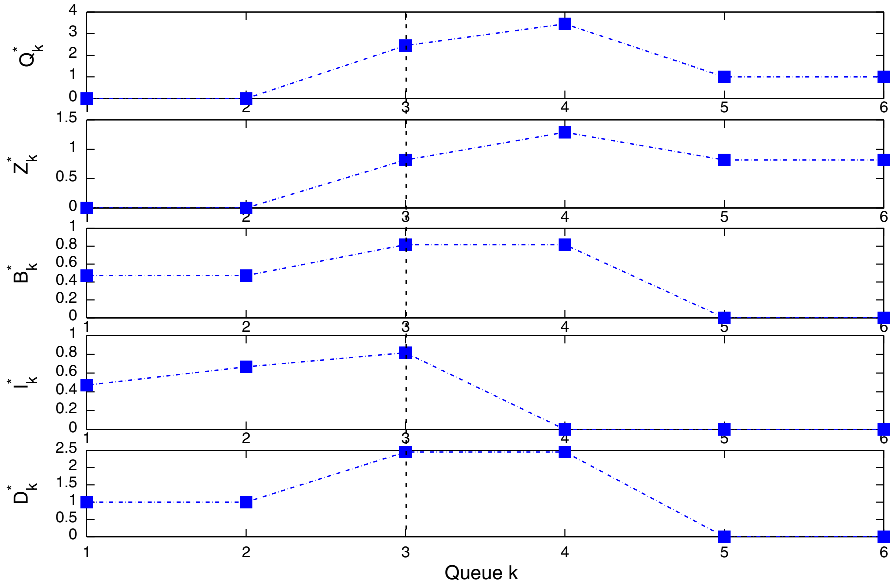

Fig. 4 LIL limits of Example 2 as functions of k, $1 \le k \le 6$, with $k_0 = 3$, $\sum_{k=1}^{3} \rho_k = 1$ and $\sum_{k=1}^{4} \rho_k > 1$

More numerical examples

A type-1 OL example

Example 3 (Discontinuities of the LIL limits in the class index) Consider a 6-queue example $\lambda_k = 1$, $\mu_k = 3$, $c_{a,k} = c_{s,k} = 1$ for all $1 \le k \le K = 6$. This example belongs to the type-1 OL case because $k_0 = 3$, $\sum_{k=1}^{3} \rho_k = 1$ and $\sum_{k=1}^{4} \rho_k = 4/3 > 1$.

According to Theorem 4, we compute the LIL limits in Table 2 and plot these limits as functions of k in Fig. 4. The vertical line in 4 serving as a "benchmark" for $\rho = 1$ (because $\sum_{k=1}^{3} \rho_k = 1$ and $\sum_{k=1}^{4} \rho_k > 1$). We see that all LIL limits jump at k_0 and $k_0 + 1$. The LIL limits Q_k^*, Z_k^*, B_k^*, and D_k^* all peak at $k_0 = 3$ and $k_0 + 1 = 4$, where stochastic processes Q_k, Z_k, B_k, and D_k experiencing the largest asymptotical stochastic variability. The LIL I_k^* increases in k and peak at $k = k_0 = 3$, it then drops to 0. This is so because the variability of I_k is cumulative (thus increasing) for $1 \le k \le k_0$ and then I_k becomes asymptotically negligible for all $k_0 < k \le K$. See Remark 5 for more discussions.

We plot the LIL limits as functions of k in Fig. 4.

LIL formulas for Example 2

We next provide the explicit LIL limits for Example 2. These limits are piecewise functions. The LIL limits for Q are

$$
\begin{aligned}
&Q_1^*(\rho_2) = 0, \quad \text{for all} \quad 0 \le \rho_2 \le 1,\\
&Q_2^*(\rho_2) = \sqrt{2} \cdot \mathbf{1}_{\{\rho_2=0.5\}} + \sqrt{1+\sqrt{0.5}+\rho_2} \cdot \mathbf{1}_{\{0.5<\rho_2\le 1\}},\\
&Q_3^*(\rho_2) = \sqrt{2} \cdot \mathbf{1}_{\{\rho_2=0\}} + \sqrt{1.5+2\rho_2+\sqrt{0.5-\rho_2}} \cdot \mathbf{1}_{\{0<\rho_2\le 0.5\}} + \sqrt{0.5} \cdot \mathbf{1}_{\{0.5<\rho_2\le 1\}},\\
&Q_4^*(\rho_2) = \sqrt{2.5} \cdot \mathbf{1}_{\{\rho_2=0\}} + \sqrt{0.5} \cdot \mathbf{1}_{\{0<\rho_2\le 1\}}.
\end{aligned}
$$

The LIL limits for Z are

$$
\begin{aligned}
&Z_1^*(\rho_2) = 0, \quad \text{for all} \quad 0 \le \rho_2 \le 1,\\
&Z_2^*(\rho_2) = \sqrt{1+2\rho_2} \cdot \mathbf{1}_{\{0.5\le\rho_2\le 1\}},\\
&Z_3^*(\rho_2) = \sqrt{2+2\rho_2} \cdot \mathbf{1}_{\{0\le\rho_2\le 0.5\}} + \mathbf{1}_{\{0.5<\rho_2\le 1\}},\\
&Z_4^*(\rho_2) = \sqrt{3} \cdot \mathbf{1}_{\{\rho_2=0\}} + \mathbf{1}_{\{0<\rho_2\le 1\}}.
\end{aligned}
$$

The LIL limits for B are

$$
\begin{aligned}
&B_1^*(\rho_2) = 1, \quad \text{for all} \quad 0 \le \rho_2 \le 1,\\
&B_2^*(\rho_2) = \sqrt{2\rho_2} \cdot \mathbf{1}_{\{0\le\rho_2<0.5\}} + \mathbf{1}_{\{0.5\le\rho_2\le 1\}},\\
&B_3^*(\rho_2) = \sqrt{1+2\rho_2} \cdot \mathbf{1}_{\{0\le\rho_2\le 0.5\}},\\
&B_4^*(\rho_2) = \sqrt{2} \cdot \mathbf{1}_{\{\rho_2=0\}}.
\end{aligned}
$$

The LIL limits for I are

$$
\begin{aligned}
&I_1^*(\rho_2) = 1, \quad \text{for all} \quad 0 \le \rho_2 \le 1,\\
&I_2^*(\rho_2) = \sqrt{1+2\rho_2} \cdot \mathbf{1}_{\{0\le\rho_2\le 0.5\}},\\
&I_3^*(\rho_2) = \sqrt{2} \cdot \mathbf{1}_{\{\rho_2=0\}},\\
&I_4^*(\rho_2) = 0, \quad \text{for all} \quad 0 \le \rho_2 \le 1.
\end{aligned}
$$

The LIL limits for D are

$$
\begin{aligned}
&D_1^*(\rho_2) = \sqrt{0.5}, \quad \text{for all} \quad 0 \le \rho_2 \le 1,\\
&D_2^*(\rho_2) = \sqrt{\rho_2} \cdot \mathbf{1}_{\{0\le\rho_2<0.5\}} + \sqrt{1.5} \cdot \mathbf{1}_{\{\rho_2=0.5\}} + \sqrt{1+\sqrt{0.5}} \cdot \mathbf{1}_{\{0.5<\rho_2\le 1\}},\\
&D_3^*(\rho_2) = \sqrt{1.5} \cdot \mathbf{1}_{\{\rho_2=0\}} + \sqrt{1+2\rho_2+\sqrt{0.5-\rho_2}} \cdot \mathbf{1}_{\{0<\rho_2<0.5\}} + \sqrt{2} \cdot \mathbf{1}_{\{\rho_2=0.5\}},\\
&D_4^*(\rho_2) = \sqrt{2} \cdot \mathbf{1}_{\{\rho_2=0\}}.
\end{aligned}
$$

More discussion on Corollary 1

Using the analytic solutions to (13), we transform the results of Corollary 1 to more detailed formulas. We consider two cases: (1) $\rho < 1$ and (2) there exists $k_0 : 1 < k_0 < K$ such that $\sum_{l=1}^{k_0} \rho_l \le 1$ and $\sum_{l=1}^{k_0+1} \rho_l > 1$.

Case 1. If $\rho \le 1$, then $\bar{Q}_k(t) = 0$ and $\bar{B}_k(t) = \rho_k t$ for $k = 1, 2, \ldots, K$. Hence (29) and (30) are the following

$$\widetilde{X}_k(t) \stackrel{\mathrm{d}}{=} \mu_k \left(\sum_{l=1}^{k} \rho_l - 1 \right) t - \sum_{l=1}^{k-1} \frac{\mu_k}{\mu_l} \widetilde{Q}_l(t) + \sum_{l=1}^{k} \frac{\mu_k}{\mu_l} \left[\lambda_l^{1/2} c_{a,l} W_{a,l}(t) - \lambda_l^{1/2} c_{s,l} W_{s,l}(t) \right], \tag{60}$$

$$\bar{B}_k(t) - \widetilde{B}_k(t) \stackrel{\mathrm{d}}{=} \rho_k t - \widetilde{B}_k(t) = \frac{1}{\mu_k} \widetilde{Q}_k(t) - \frac{1}{\mu_k} \left[\lambda_k^{1/2} c_{a,k} W_{a,k}(t) - \lambda_k^{1/2} c_{s,k} W_{s,k}(t) \right]. \tag{61}$$

Case 2. If there exists $1 < k_0 < K$ such that $\sum_{l=1}^{k_0} \rho_l \le 1$ and $\sum_{l=1}^{k_0+1} \rho_l > 1$, then $\bar{Q}_k(t) = 0$ and $\bar{B}_k(t) = \rho_k t$ for $k = 1, 2, \ldots, k_0$, and (60) and (61) hold for $1, 2, \ldots, k_0$. For $k = k_0 + 1$, we note that $\bar{Q}_{k_0+1}(t) = \bar{X}_{k_0+1}(t) = \mu_{k_0+1}(\sum_{l=1}^{k_0+1} \rho_l - 1)t$ and $\bar{B}_{k_0+1}(t) = (1 - \sum_{l=1}^{k_0} \rho_l)t$. Hence,

$$\widetilde{X}_{k_0+1}(t) \stackrel{\mathrm{d}}{=} \mu_{k_0+1} \left(\sum_{l=1}^{k_0+1} \rho_l - 1 \right) t - \sum_{l=1}^{k_0} \frac{\mu_{k_0+1}}{\mu_l} \left[\widetilde{Q}_l(t) + \lambda_l^{1/2} c_{a,l} W_{a,l}(t) - \lambda_l^{1/2} c_{s,l} W_{s,l}(t) \right] + \left[\lambda_{k_0+1}^{1/2} c_{a,k_0+1} W_{a,k_0+1}(t) - \mu_{k_0+1}^{1/2} c_{s,k_0+1} \sqrt{1 - \sum_{l=1}^{k_0} \rho_i W_{s,k_0+1}(t)} \right], \tag{62}$$

and

$$\begin{aligned} &\bar{B}_{k_0+1}(t) - \widetilde{B}_{k_0+1}(t) \\ &= \left(1 - \sum_{l=1}^{k_0} \rho_l \right) t - \widetilde{B}_{k_0+1}(t) \\ &\stackrel{\mathrm{d}}{=} \frac{1}{\mu_{k_0+1}} \left[\widetilde{Q}_{k_0+1}(t) - \mu_{k_0+1} \left(\sum_{l=1}^{k_0+1} \rho_l - 1 \right) t \right] \end{aligned}$$

$$-\frac{1}{\mu_{k_0+1}}\left[\lambda_{k_0+1}^{1/2}c_{a,k_0+1}W_{a,k_0+1}(t)-\mu_{k_0+1}^{1/2}c_{s,k_0+1}\sqrt{1-\sum_{l=1}^{k_0}\rho_i W_{s,k_0+1}(t)}\right]. \tag{63}$$

For $k=k_0+i$, $i=2,3,\ldots,K-k_0$, $\bar{Q}_{k_0+i}(t)=\lambda_{k_0+i}t$, $\bar{B}_{k_0+i}(t)=0$, we have

$$\begin{aligned}\widetilde{X}_{k_0+i}(t) &\stackrel{\mathrm{d}}{=} \lambda_{k_0+i}t-\sum_{l=1}^{k_0}\frac{\mu_{k_0+i}}{\mu_l}\widetilde{Q}_l(t)-\frac{\mu_{k_0+i}}{\mu_{k_0+1}}\left[\widetilde{Q}_{k_0+1}(t)-\mu_{k_0+1}\left(\sum_{l=1}^{k_0+1}\rho_l-1\right)t\right]\\ &-\sum_{l=k_0+2}^{k_0+i-1}\frac{\mu_{k_0+i}}{\mu_l}\left[\widetilde{Q}_l(t)-\lambda_l t\right]+\sum_{l=1}^{k_0}\frac{\mu_{k_0+i}}{\mu_l}\left[\lambda_l^{1/2}c_{a,l}W_{a,l}(t)-\lambda_l^{1/2}c_{s,l}W_{s,l}(t)\right]\\ &+\frac{\mu_{k_0+i}}{\mu_{k_0+1}}\left[\lambda_{k_0+1}^{1/2}c_{a,k_0+1}W_{a,k_0+1}(t)-\mu_{k_0+1}^{1/2}c_{s,k_0+1}\sqrt{1-\sum_{l=1}^{k_0}\rho_i W_{s,k_0+1}(t)}\right]\\ &+\sum_{l=k_0+2}^{k_0+i}\frac{\mu_{k_0+i}}{\mu_l}\lambda_l^{1/2}c_{a,l}W_{a,l}(t),\end{aligned} \tag{64}$$

and

$$\bar{B}_{k_0+i}(t)-\widetilde{B}_{k_0+i}(t)\stackrel{\mathrm{d}}{=}\frac{\widetilde{Q}_{k_0+i}(t)-\lambda_{k_0+i}t}{\mu_{k_0+i}}-\frac{\lambda_{k_0+i}^{1/2}c_{a,k_0+i}W_{a,k_0+i}(t)}{\mu_{k_0+i}}. \tag{65}$$

Proof of Lemma 2

According to (13), $\bar{B}_k(t)=\rho_k t$ and $\bar{X}_k(t)=-\theta_k t<0$ for $k=1,2,\ldots,K$. Next, (27) implies that $\widetilde{Q}_1(t)=\Phi(\widetilde{X}_1)(t)$ is a reflected BM, where $\widetilde{X}_1(t)=\bar{X}_1(t)+W_1(t)$ is a BM with negative drift $-\theta_1$ and variance parameter $\mu_1\sigma_1$. Theorem 6.2 in [18] implies that

$$\mathsf{P}\left\{\sup_{0\le t\le T}\widetilde{Q}_1(t)\ge z\right\}\le\exp\left\{-\frac{2\theta_1}{\mu_1^2\sigma_1^2}z\right\}\le\exp\{-2\gamma z\},\quad z\ge 0. \tag{66}$$

We next consider $k=2,3,\ldots,K$. For $z\ge 0$,

$$\begin{aligned}&\mathsf{P}\left\{\sup_{0\le t\le T}\widetilde{Q}_k(t)\ge z\right\}\\ &=\mathsf{P}\left\{\sup_{0\le t\le T}\left\{\widetilde{X}_k(t)+\sup_{0\le s\le t}\left[-\widetilde{X}_k(s)\right]\right\}\ge z\right\}\\ &=\mathsf{P}\left\{\sup_{0\le t\le T}\sup_{0\le s\le t}\left[\widetilde{X}_k(t)-\widetilde{X}_k(s)\right]\ge z\right\}\end{aligned}$$

$$\leq \mathsf{P}\left\{\sup_{0\leq t\leq T}\sup_{0\leq s\leq t}\left[(\bar{X}_k(t)+W_k(t))-(\bar{X}_k(s)+W_k(s))\right]\geq\frac{z}{2}\right\}$$
$$+\mathsf{P}\left\{\sup_{0\leq t\leq T}\sup_{0\leq s\leq t}\left[\sum_{l=1}^{k-1}\frac{\mu_k}{\mu_l}\widetilde{Q}_l(s)-\sum_{l=1}^{k-1}\frac{\mu_k}{\mu_l}\widetilde{Q}_l(t)\right]\geq\frac{z}{2}\right\}$$
$$\leq \mathsf{P}\left\{\sup_{0\leq s\leq T}[\bar{X}_k(s)+W_k(s)]\geq\frac{z}{2}\right\}+\mathsf{P}\left\{\sup_{0\leq s\leq T}\sum_{l=1}^{k-1}\frac{\mu_k}{\mu_l}\widetilde{Q}_l(s)\geq\frac{z}{2}\right\}, \tag{67}$$

where the first equality holds because $\widetilde{X}_k(0)=0$ and the first inequality holds by (29). To bound the second term in (67), we have

$$\mathsf{P}\left\{\sup_{0\leq s\leq T}\sum_{l=1}^{k-1}\frac{\mu_k}{\mu_l}\widetilde{Q}_l(s)\geq\frac{z}{2}\right\}\leq\sum_{l=1}^{k-1}\mathsf{P}\left\{\sup_{0\leq s\leq T}\widetilde{Q}_l(s)\geq\frac{\min\{\mu_1,\ldots,\mu_{k-1}\}}{(k-1)\mu_k}\cdot\frac{z}{2}\right\}$$
$$\leq\sum_{l=1}^{k-1}\mathsf{P}\left\{\sup_{0\leq s\leq T}\widetilde{Q}_l(s)\geq\delta_k z\right\} \tag{68}$$

with δ_k given in Lemma 2.

We are now ready to prove (33) for $2\leq k\leq K$. We use induction. First, when $k=2$, using (67), (68) and the fact that $\delta_k\leq\frac{1}{2}$, we have

$$\mathsf{P}\left\{\sup_{0\leq t\leq T}\widetilde{Q}_2(t)\geq z\right\}\leq\mathsf{P}\left\{\sup_{0\leq s\leq T}[\bar{X}_2(s)+W_2(s)]\geq\delta_2 z\right\}$$
$$+\mathsf{P}\left\{\sup_{0\leq s\leq T}\widetilde{Q}_1(s)\geq\delta_2 z\right\}$$
$$\leq\exp\left\{-\frac{2\theta_2}{\mu_2^2\sigma_2^2}\delta_2 z\right\}+\exp\left\{-\frac{2\theta_1}{\mu_1^2\sigma_1^2}\delta_2 z\right\}\leq N_2\exp\left\{-2\gamma\delta_2 z\right\},$$

where the second inequality holds by (66) and Lemma 5.5 in [18] (with $\bar{X}_2(t)+W_2(t)$ being a BM with negative drift $-\theta_2$ and variance parameter $\mu_2\sigma_2$).

Next, assume (33) holds for classes $2,\ldots,k$. For class $k+1$, we have

$$\mathsf{P}\left\{\sup_{0\leq t\leq T}\widetilde{Q}_{k+1}(t)\geq z\right\}$$
$$\leq\mathsf{P}\left\{\sup_{0\leq s\leq T}[\bar{X}_{k+1}(t)+W_{k+1}(s)]\geq\delta_{k+1}z\right\}+\sum_{l=1}^{k}\mathsf{P}\left\{\sup_{0\leq s\leq T}\widetilde{Q}_l(s)\geq\delta_{k+1}z\right\}$$
$$\leq\exp\left\{-\frac{2\theta_{k+1}\delta_{k+1}z}{\mu_{k+1}^2\sigma_{k+1}^2}\right\}+\exp\left\{-\frac{2\theta_1\delta_{k+1}z}{\mu_1^2\sigma_1^2}\right\}+\sum_{l=2}^{k}N_l\exp\left\{-2\gamma\delta_2\delta_3\cdots\delta_l\delta_{k+1}z\right\}$$

$$\leq N_{k+1} \exp\{-2\gamma\delta_2\delta_3\cdots\delta_{k+1}z\} = N_{k+1}\ \exp\left\{-2\gamma\prod_{j=1}^{k+1}\delta_j\, z\right\},$$

where the first inequality holds by (67), (50), and the fact that $\delta_k \leq 1/2$, and the second inequality holds by the induction hypothesis and Lemma 5.5 in [18] (with $\bar{X}_{k+1}(t) + W_{k+1}(t)$ being a BM with negative drift $-\theta_{k+1}$ and variance parameter $\mu_{k+1}\sigma_{k+1}$). □

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